A Simple Sub-Polynomial Degree Coboundary Expander

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The Plan

1. Simplicial Complex 'high dimensional graphs'
   (a) What are they?
   (b) Expansion on simplicial complexes.

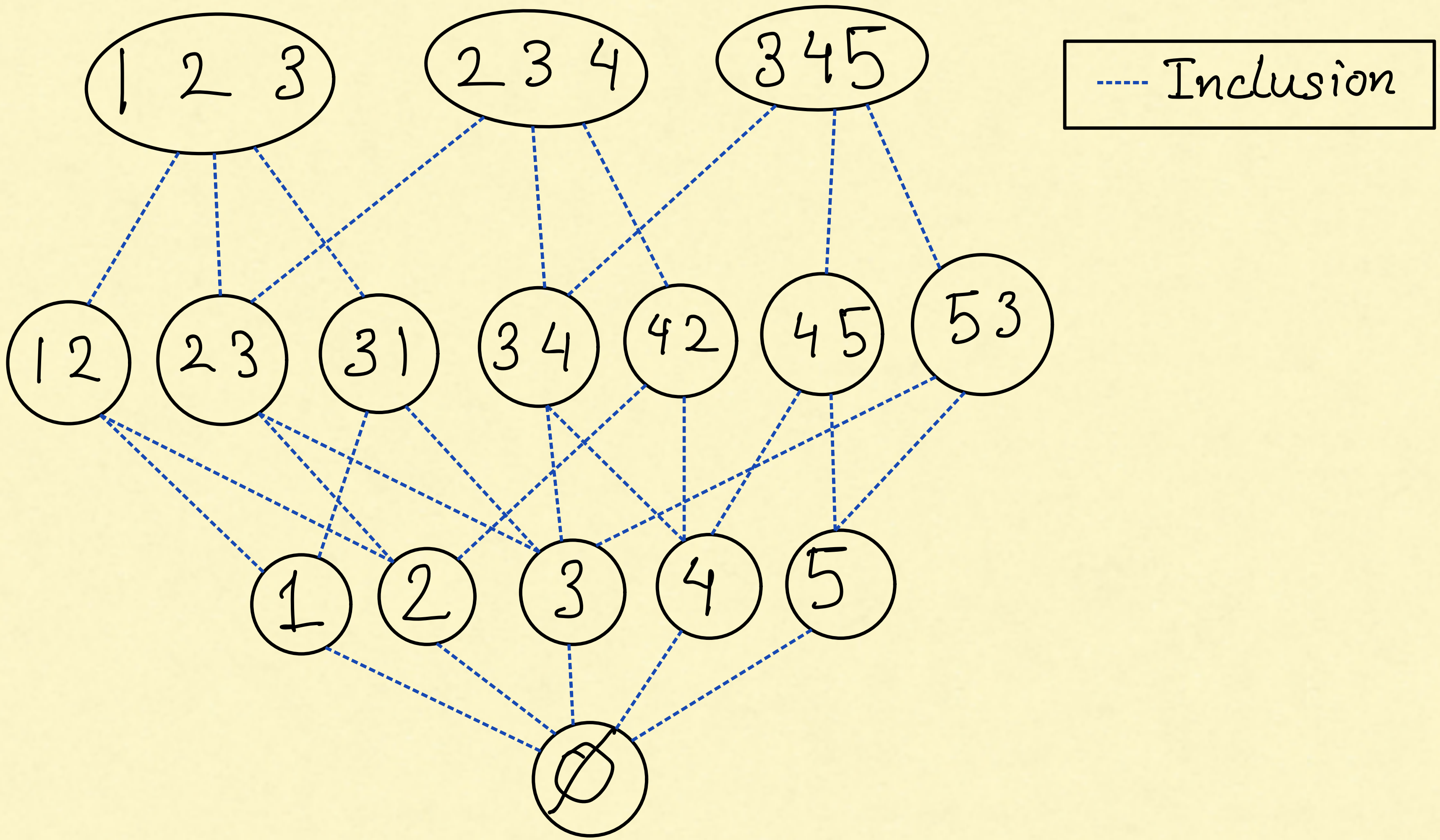
2. Our Construction

3. Some applications
   (a) Label Cover Hardness
   (b) Agreement tests

Simplicial Complexes

1 2 3   2 3 4   3 4 5

Simplicial Complexes

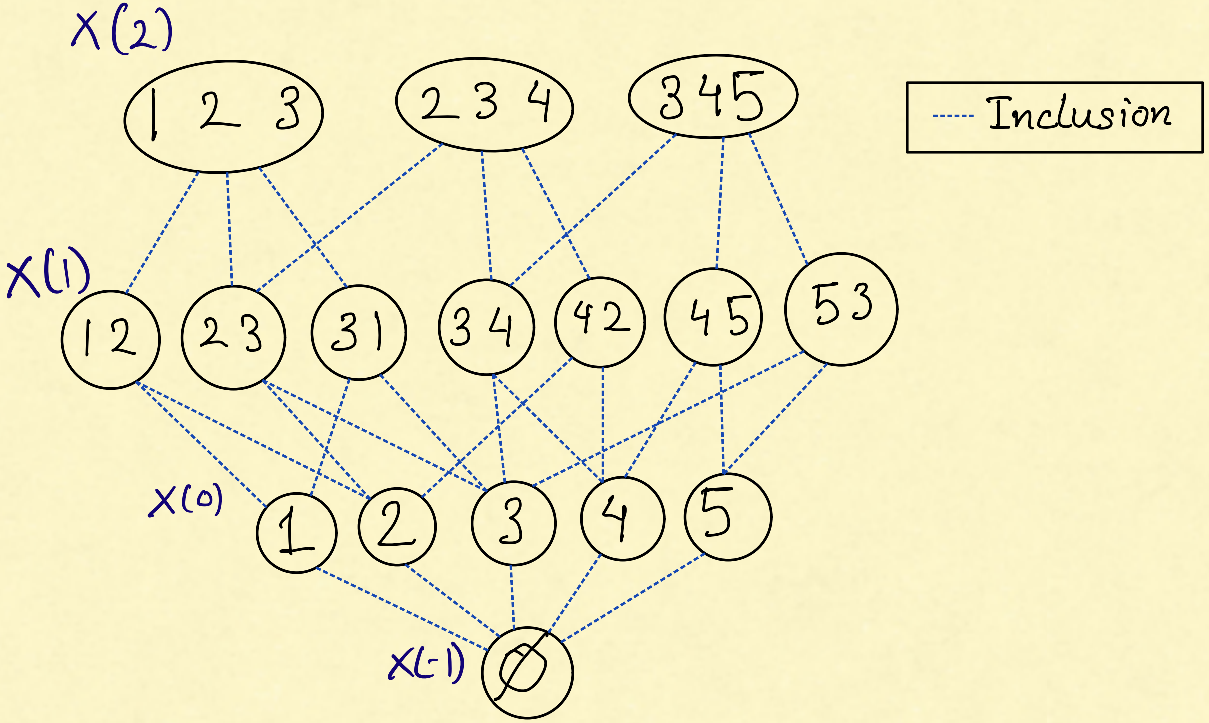


Simplicial Complexes

A Simplicial Complex

X = Downward Closed Set System

$X(\ell) = \ell + 1$ sized "faces"



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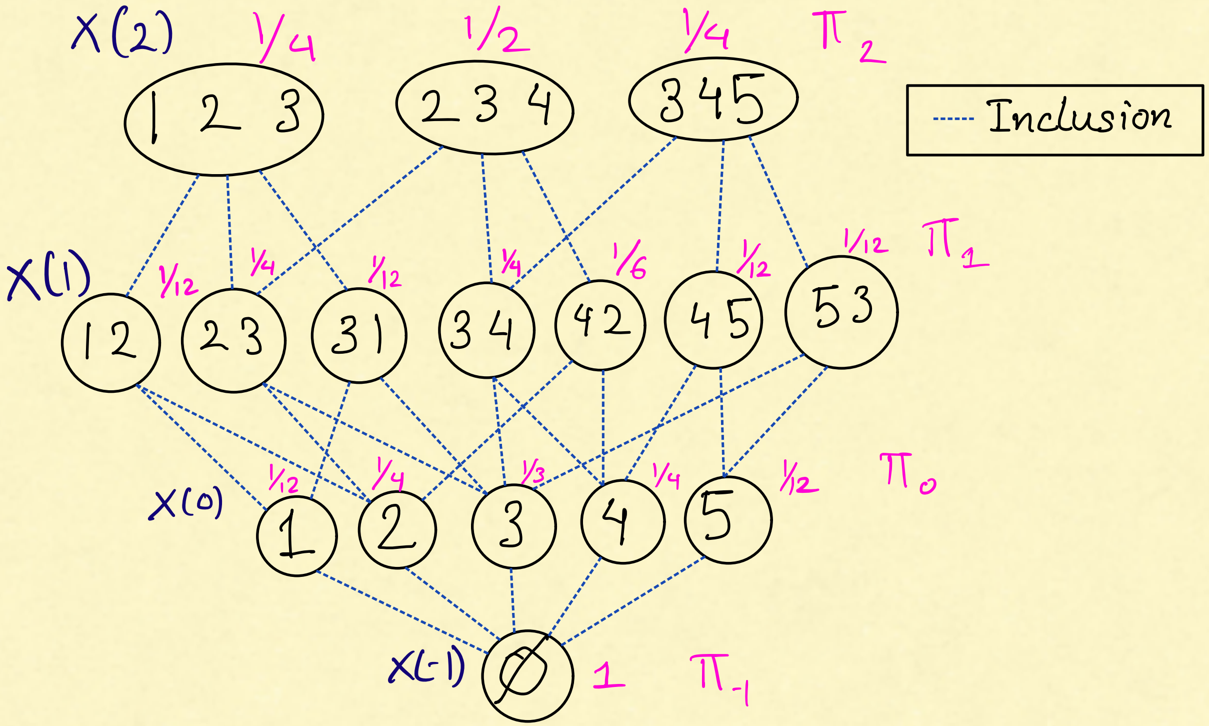


$X(\ell) = \ell + 1$ sized "faces"

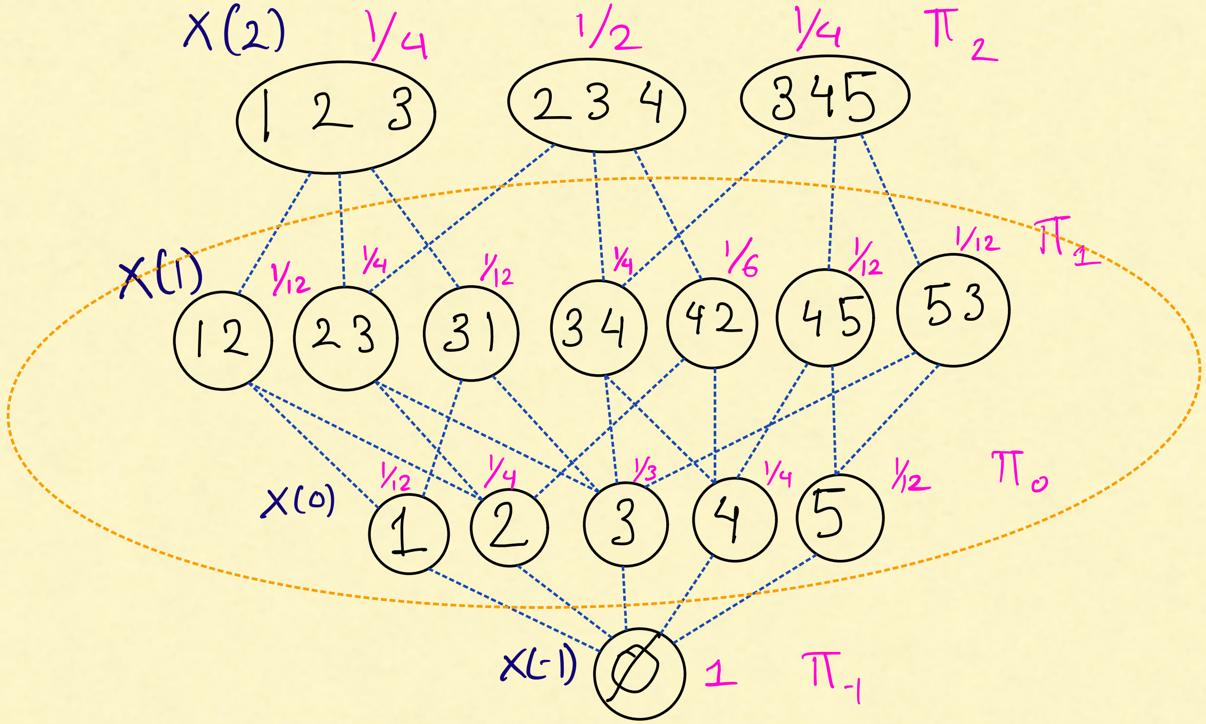
Π_k: Some arbitrary distribution on top faces.

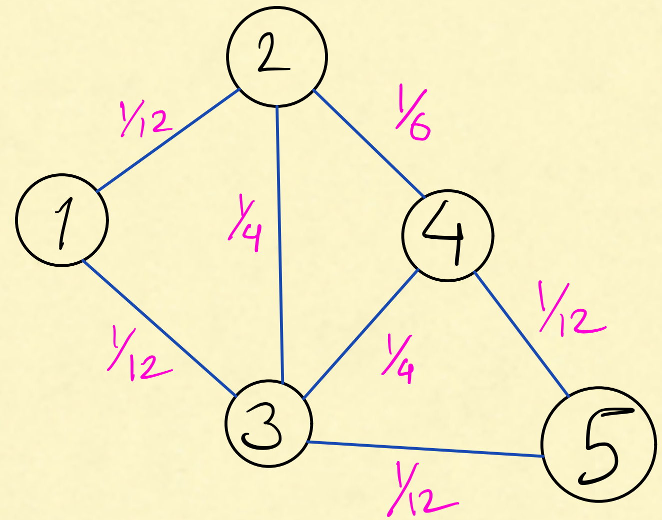
Π_ℓ: (i) sample $s \sim \Pi_k$. (ii) Sample $t \subseteq s$ of size $\ell+1$ uar.

Expansion on Simplicial Complexes

1. Local-Spectral Expansion

   (i) The underlying graph expands





Expansion on Simplicial Complexes

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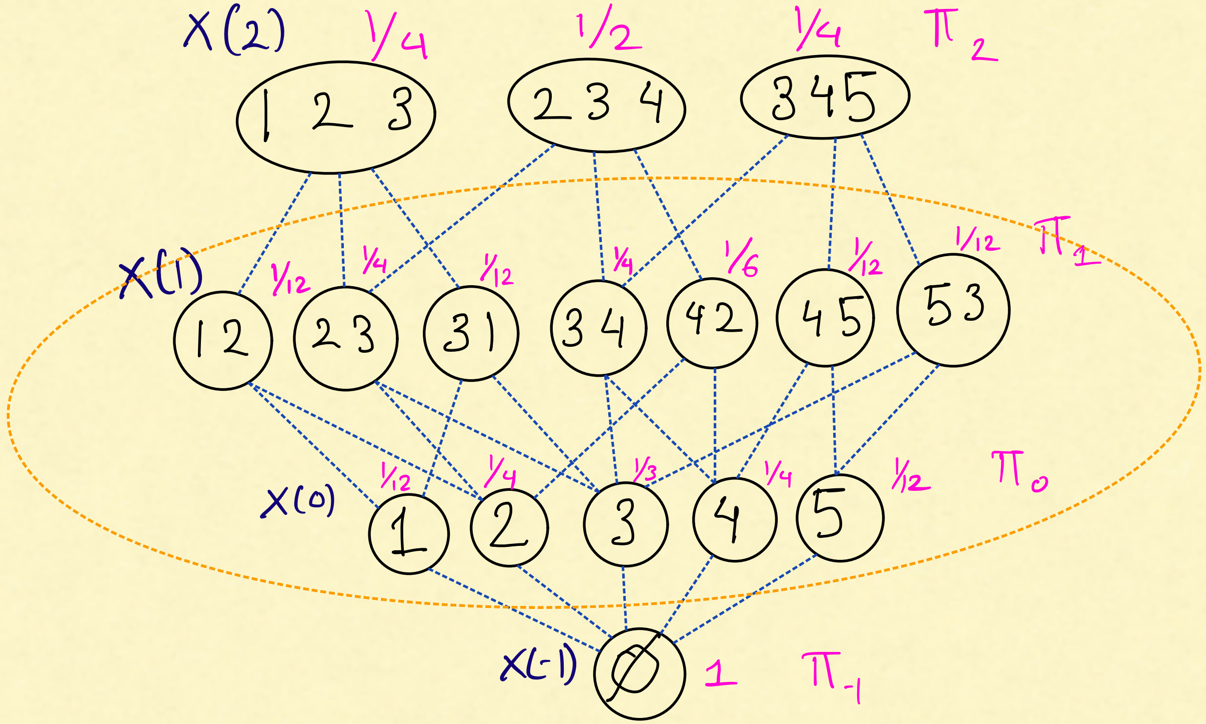
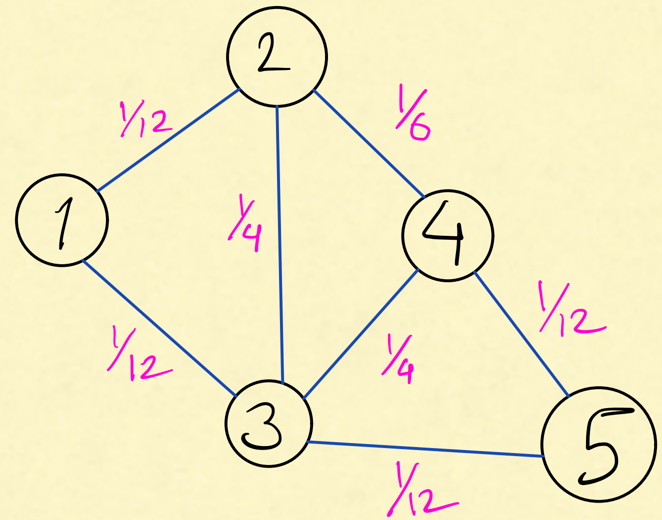
(i) The underlying graph expands $\underset{u\sim v}{\mathbb{E}}[f(u)g(v)] = \mathbb{E}[f]\,\mathbb{E}[g] \pm \lambda\sqrt{\mathrm{Var}(f)\,\mathrm{Var}(g)}$





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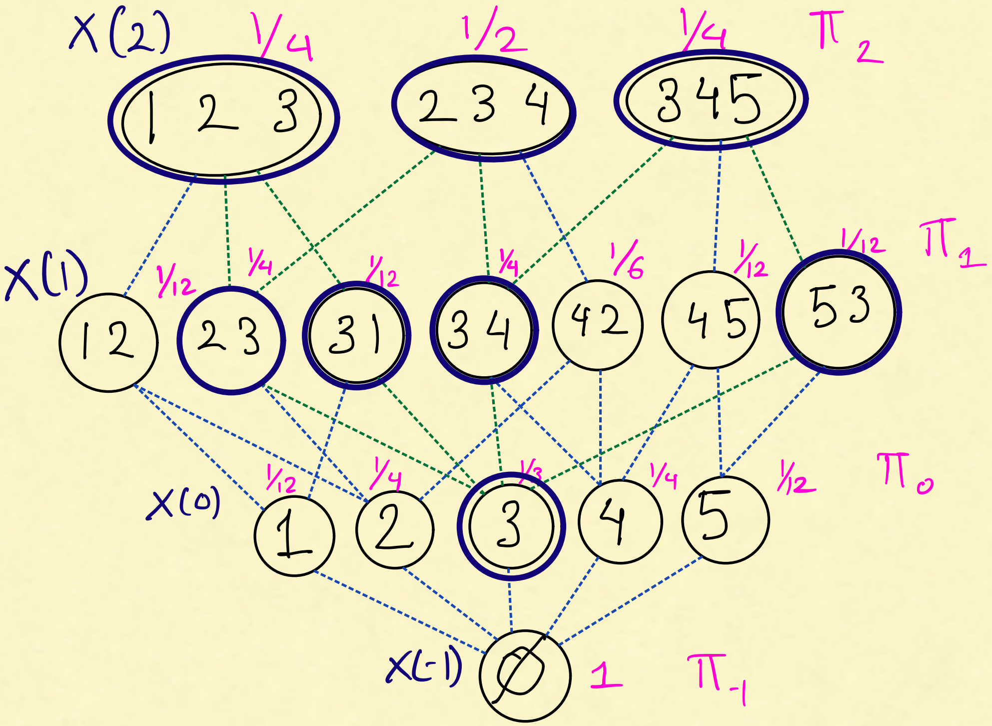
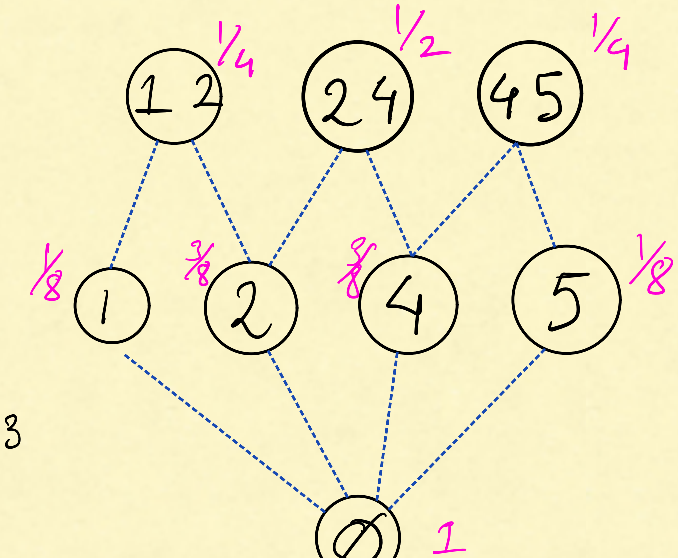


⟹
Link of 3



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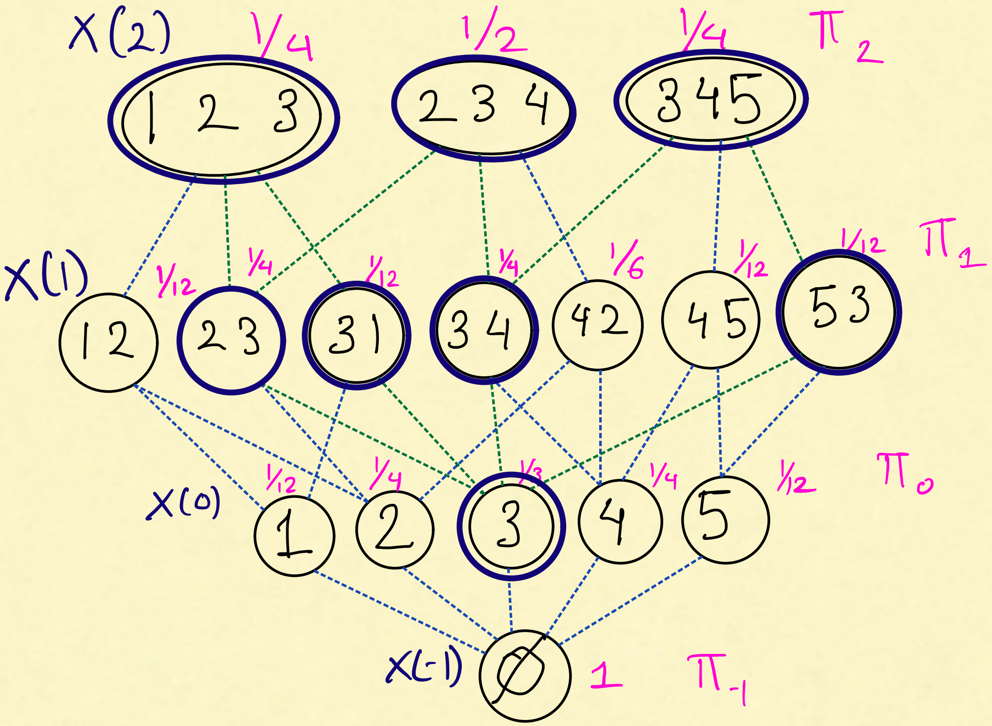

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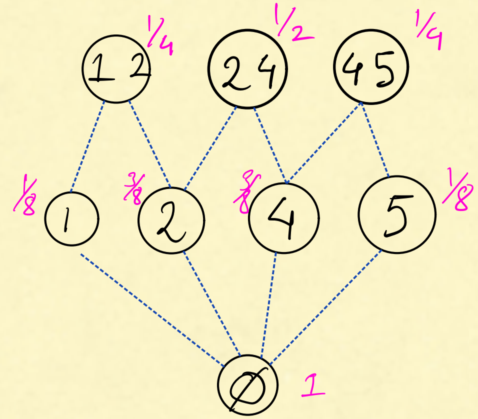

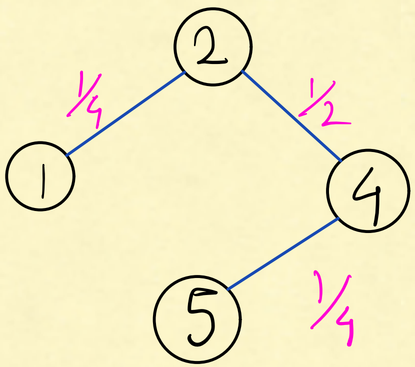

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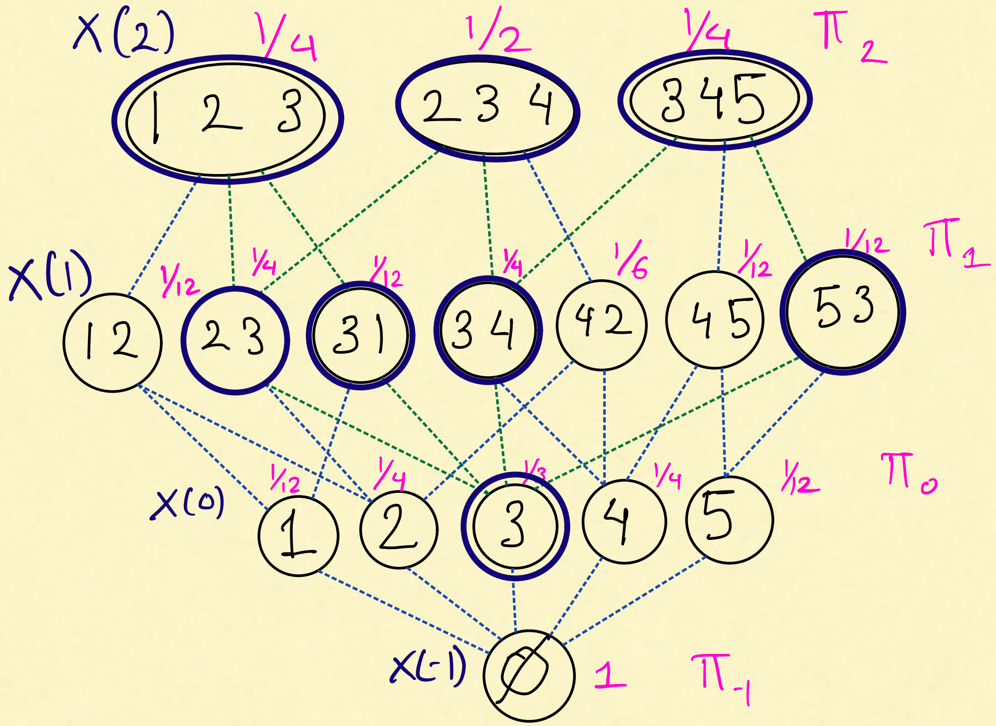

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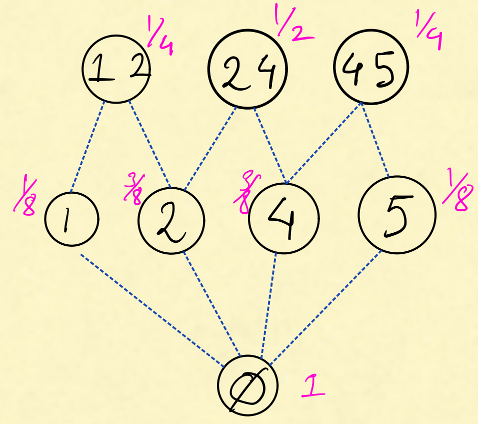

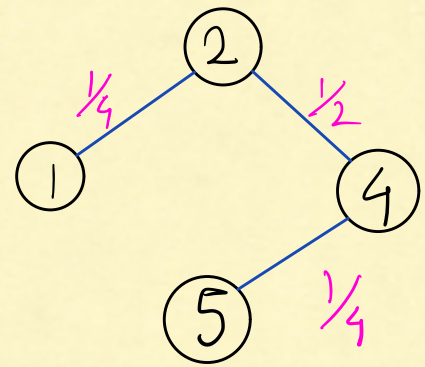

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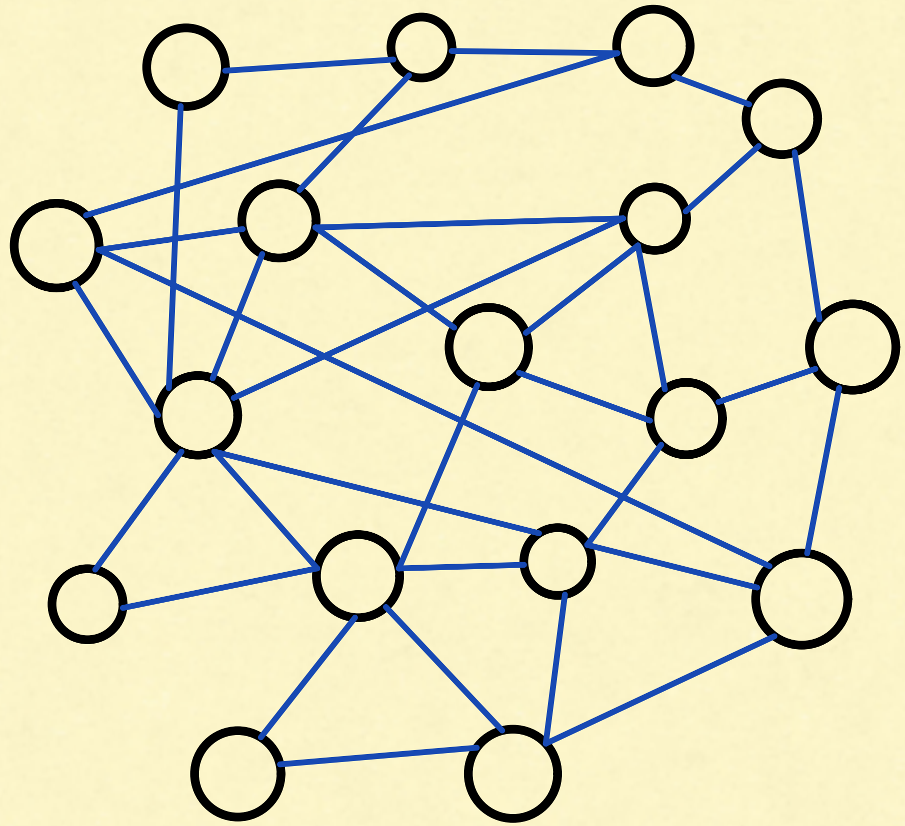

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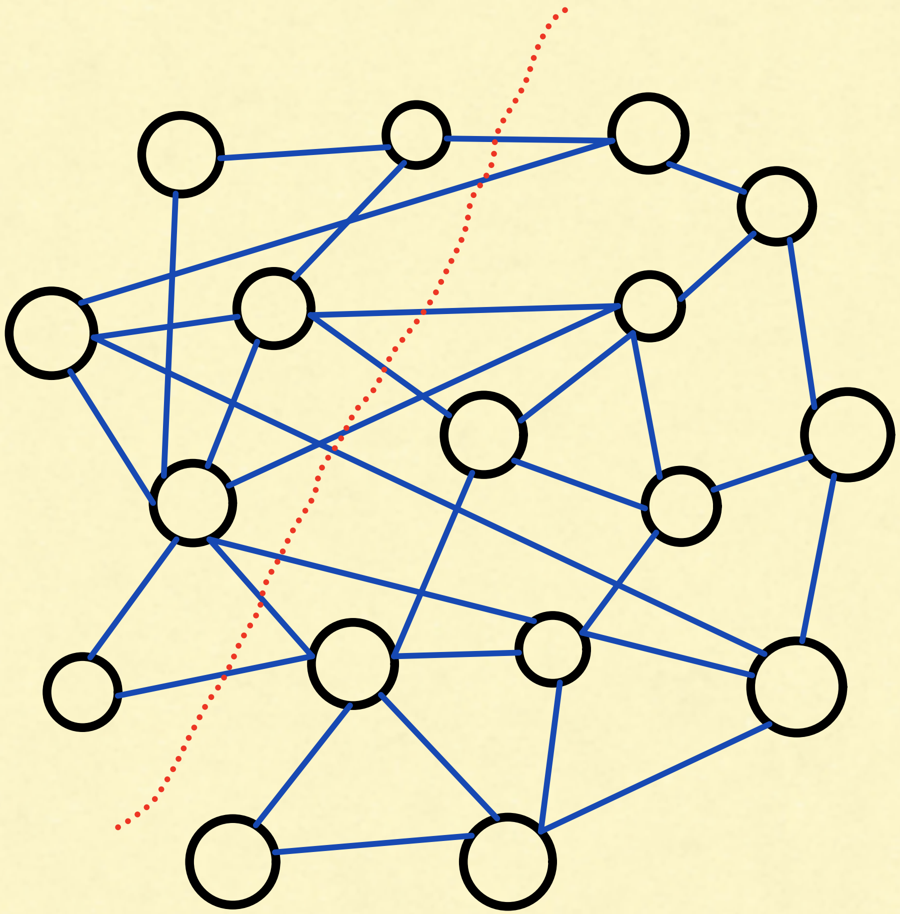

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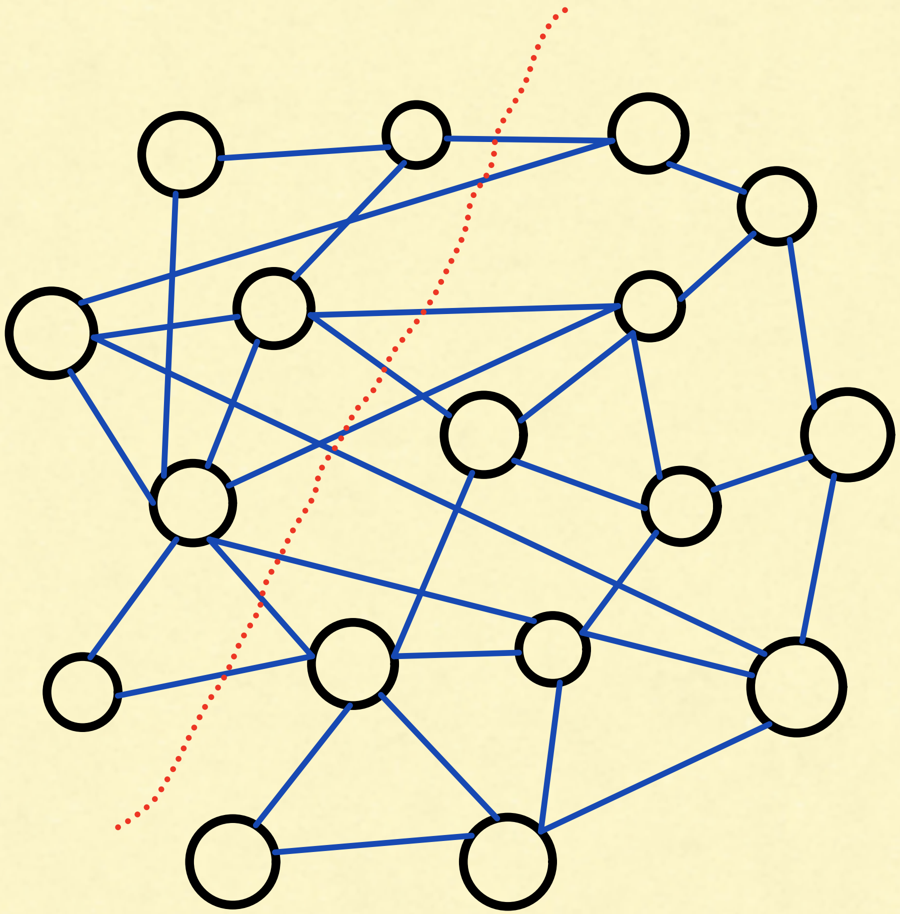
$$\phi_G(S) = \frac{wt(E(S,\bar{S}))}{\sum_{v \in S} d_v}$$



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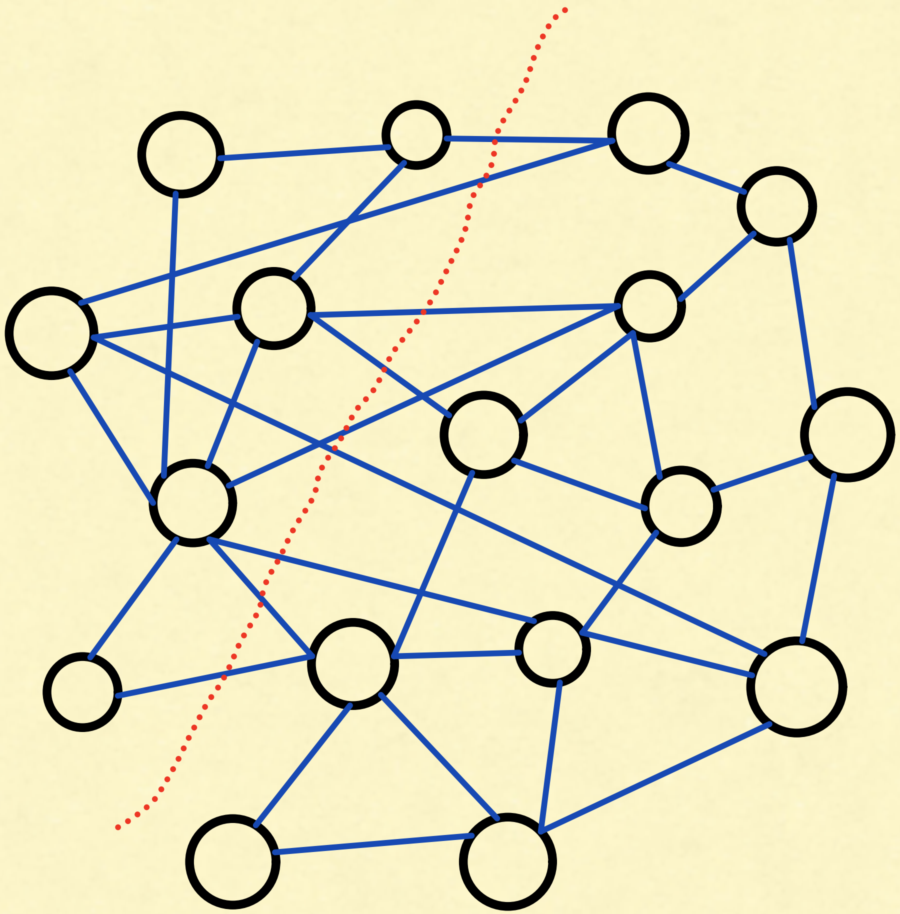

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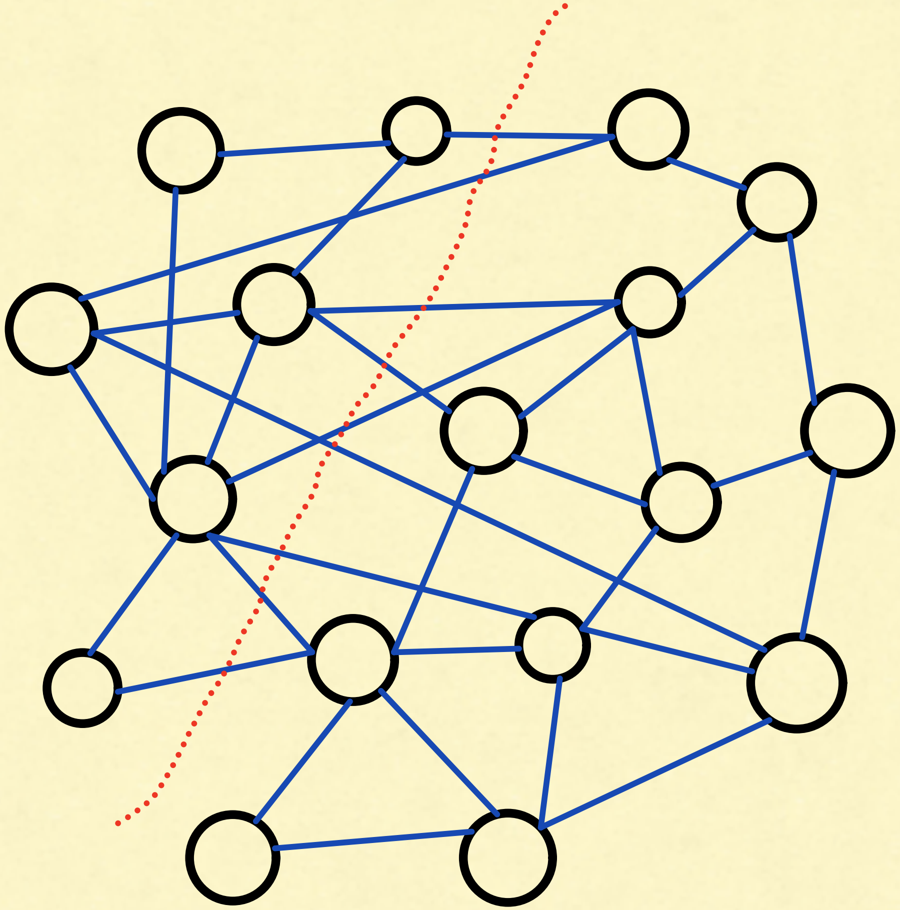
$$\phi_G = \min_S \phi_G(S)$$



CoChains

Take any set $S \subseteq X(0)$ of vertices.

$S \longleftrightarrow f_S$ where $f_S : X(0) \longrightarrow \mathbb{F}_2$

$$f_S(v) = \begin{cases} 1 & v \in S \\ 0 & \text{o.w.} \end{cases}$$

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Similarly, for a set $B \subseteq X(1)$ of edges

$B \longleftrightarrow f_B$ where $f_B : X(1) \to \mathbb{F}_2$

$$f_B(e) = \begin{cases} 1 & e \in B \\ 0 & \underline{\text{o.w.}} \end{cases}$$

CoChains

$$C^{-1} = \{ f : X(-1) \to \Gamma \} \cong \Gamma \text{ (group)}$$

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$$C^{1} = \{ f : \vec{X}(1) \to \Gamma \mid f(uv) = f(vu)^{-1} \}$$

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For $\Gamma = \mathbb{F}_2$, observe $f : X(0) \to \mathbb{F}_2 \longleftrightarrow S = \{ v \in X(0) \mid f(v) = 1 \}$

$g : X(1) \to \mathbb{F}_2 \longleftrightarrow S = \{ uv \in X(1) \mid f(uv) = 1 \}$

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cochains (Think sets)

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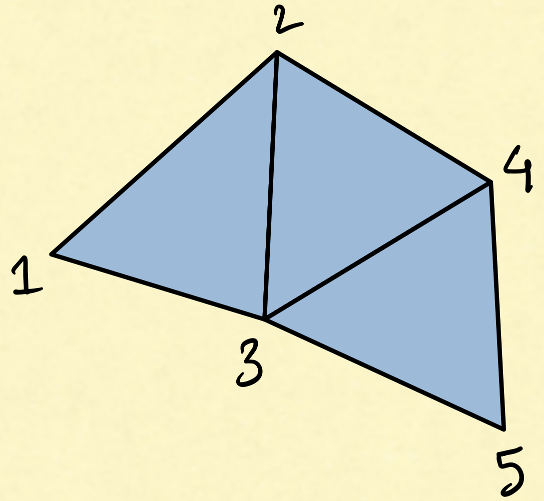

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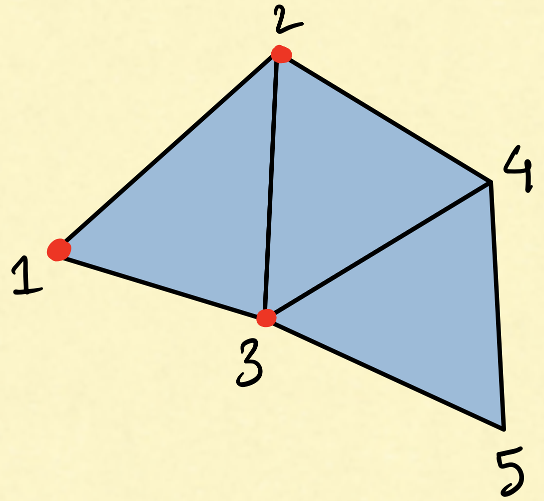

$$S = \{1, 2, 3\}$$

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$$\underbrace{24 + \cancel{12} + \cancel{23}}_{2} + \underbrace{\cancel{32} + 34 + \cancel{31}}_{3} \; + \cancel{12} + \cancel{13}$$

$\delta f s$

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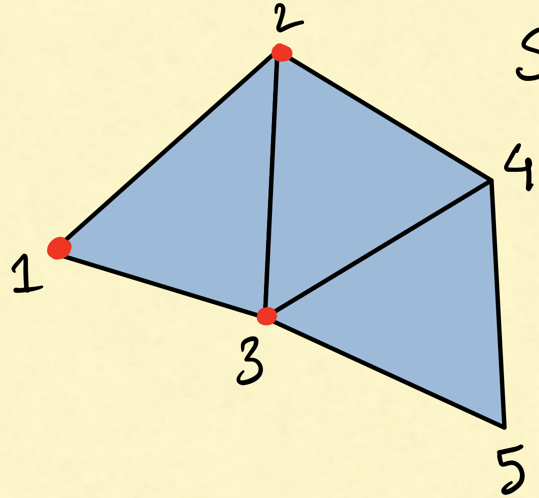

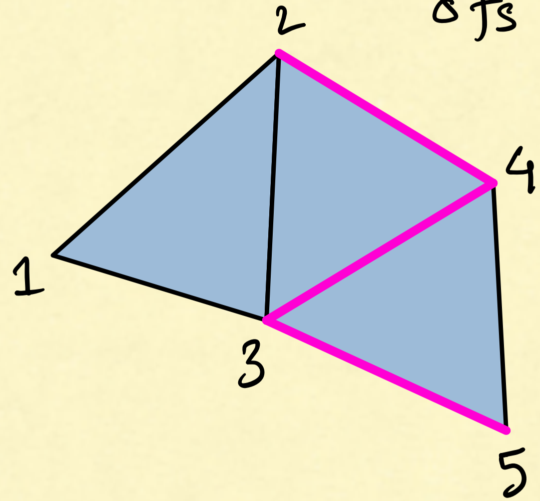

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$$dist(\mathbb{1}_S, 0) = \Pi_0(S)$$

$$dist(\mathbb{1}_S, 1) = \Pi_0(\bar{S})$$

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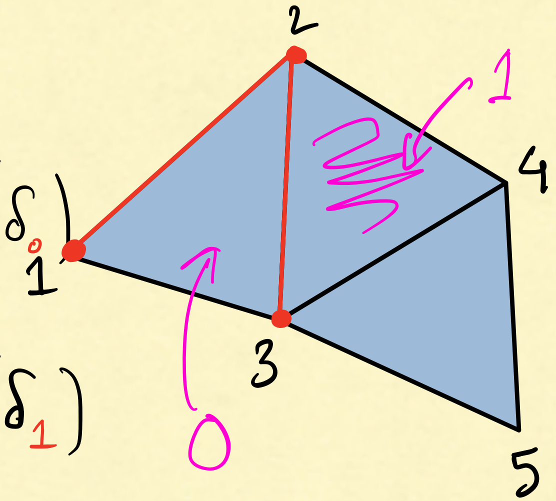

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Main Result

There is a "simple" construction of $(2k+1)$-dimensional simplicial complexes such that

(i) Degree $\lesssim 2^{O_\lambda(k\sqrt{\log N})}$

(ii) (a) Local Spectral Expansion $= \lambda$

(b) Coboundary Expansion $= \Omega(1)$.

The Construction

1. Start with $\mathbb{F}_q^d$

2. $\mathcal{Y}_q^d$ is the complex:

(i) Vertices: $\mathcal{Y}_q^d(0)$ = Non-trivial subspaces of $\mathbb{F}_q^d$

(ii) Top faces:

$$\mathcal{Y}_q^d(d-2) = \{(W_1 \subseteq W_2 \subseteq \ldots W_{d-1}) \mid \dim(W_i) = i\}$$

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$$\text{Degree} \approx q^{\Theta(d^2)} \qquad N = q^{\Theta(d^2)}$$

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$\dim(W_d) = d$

$d - \varepsilon d$ $\quad$ $\dim(W)$ $\quad$ $d + \varepsilon d$

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3. Project to $S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}$

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2. $\mathcal{Y}_q^d$ is the complex with top faces:

$$\mathcal{Y}_q^d(d-2) = \{(W_1 \subseteq W_2 \subseteq \dots W_{d-1}) \mid \dim(W_i) = i\}$$

3. Project to $S_k = \{d-k, \dots, d-1, d, d+1, \dots, d+k\}$

$$\text{Degree} = q^{\Theta(dk)} \quad N = q^{\Theta(d^2)}.$$

Faces Complex

$$F^r X(0) = X(r)$$

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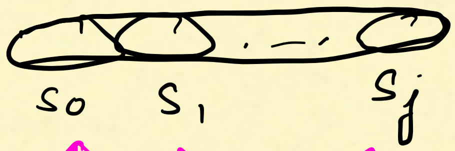
$$\{\underbrace{s_0}_{\in X(r)}, \underbrace{s_1}_{X(r)}, \dots, s_j\} \in F^r X \quad \text{iff} \quad s_0 \sqcup s_1 \sqcup s_2 \sqcup \dots \sqcup s_j \in X.$$



$$\in \underbrace{X(\sim jr)}_{j(r+1)-1}$$

Main Result

There is a "simple" construction of
(2k+1)-dimensional simplicial complexes
such that

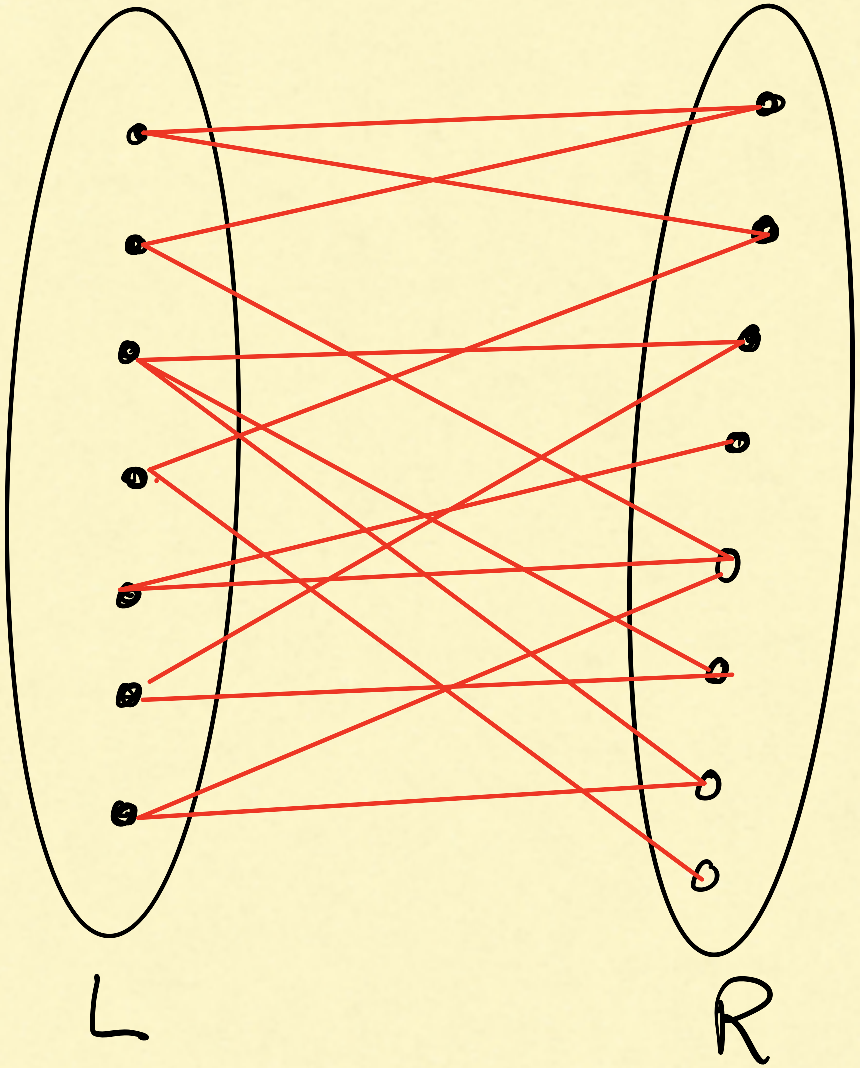
(i) Degree $\lesssim 2^{O_\lambda(k\sqrt{\log N})}$

(ii) (a) Local Spectral Expansion = λ

(b) Coboundary Expansion $= \Omega(1)$.

Moreover, coboundary Expansion of its r-faces complex $= 2^{-\tilde{O}(\sqrt{r})}$.

Label Cover Problem



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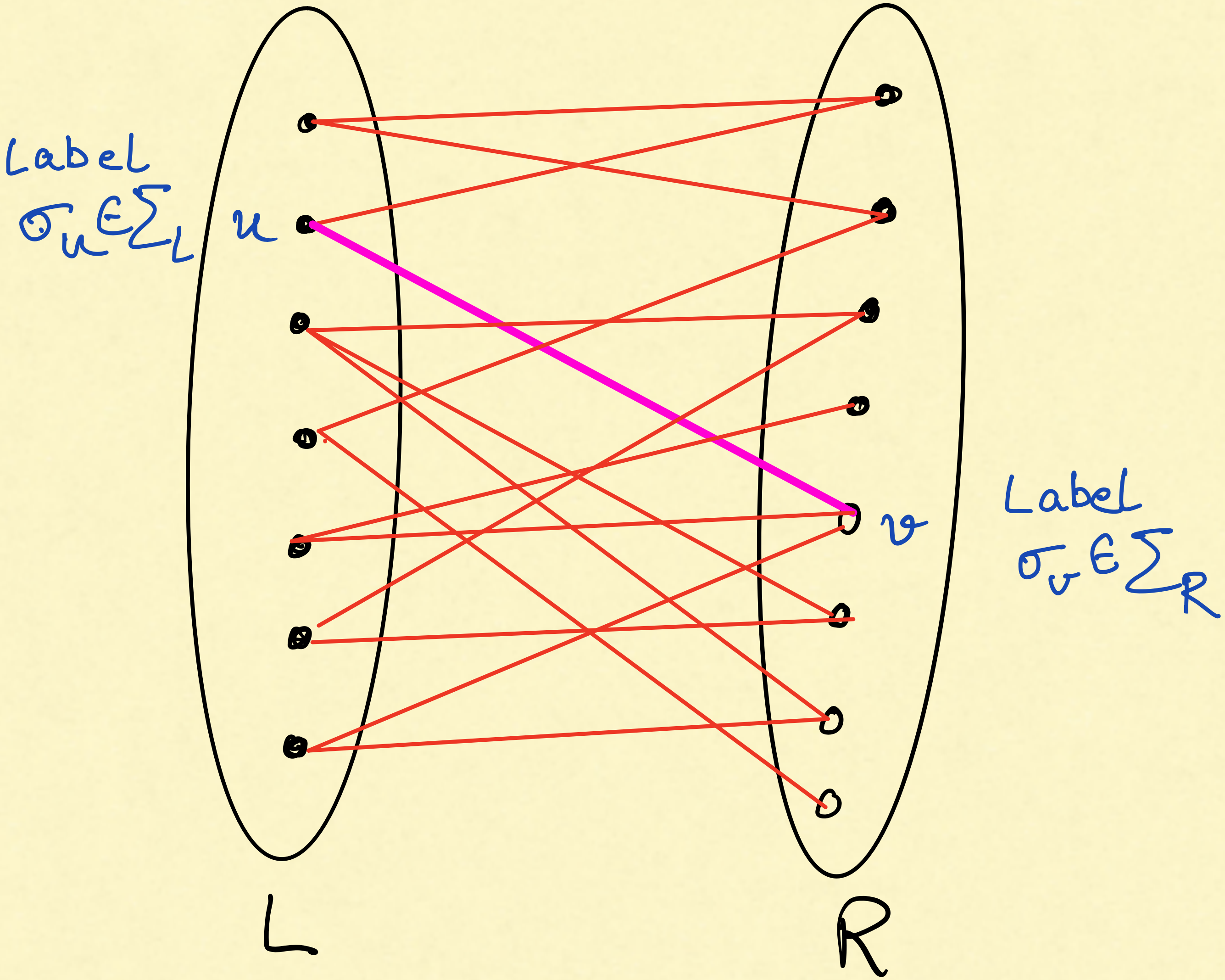

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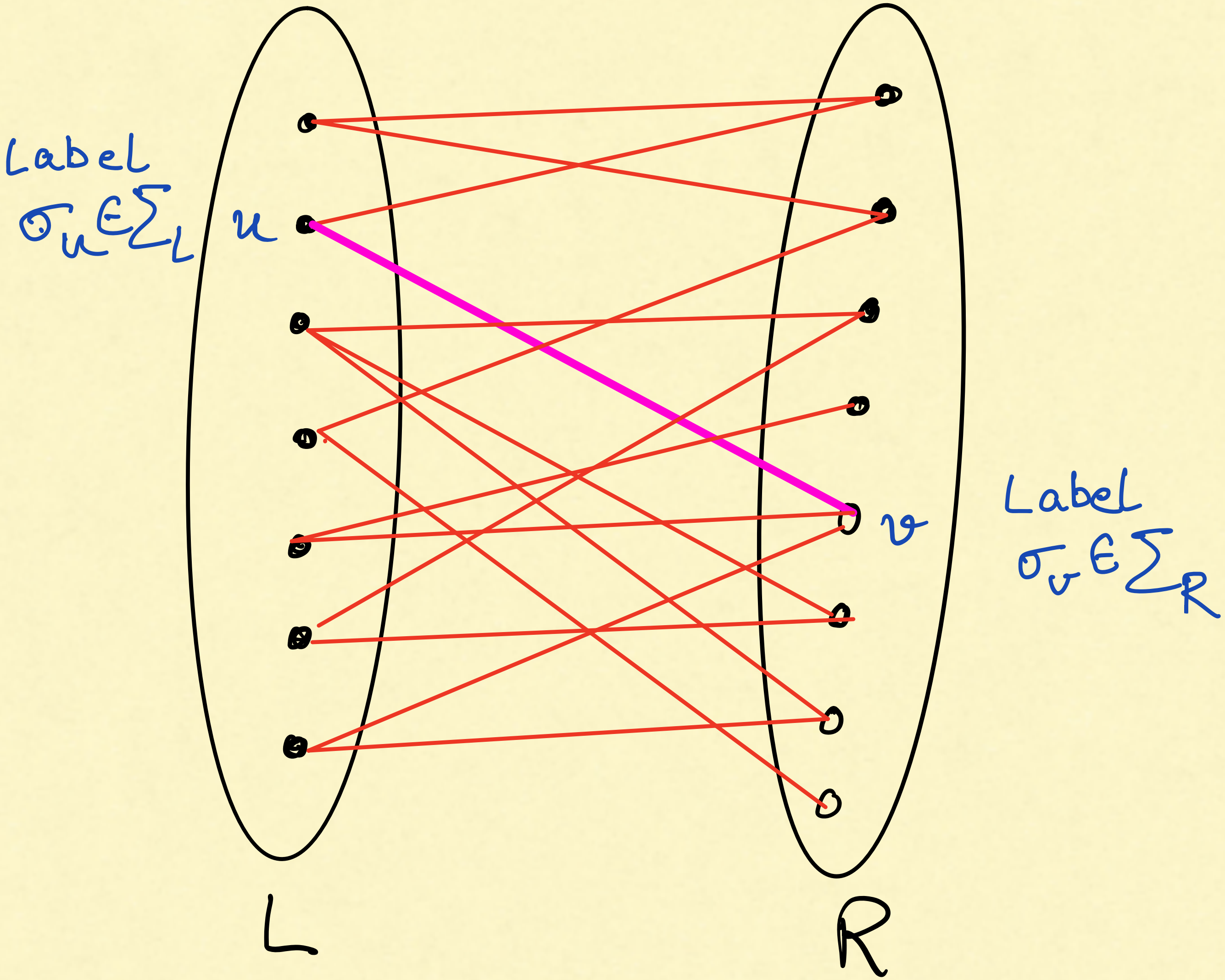

Want $\phi_{uv}(\sigma_u) = \sigma_v$

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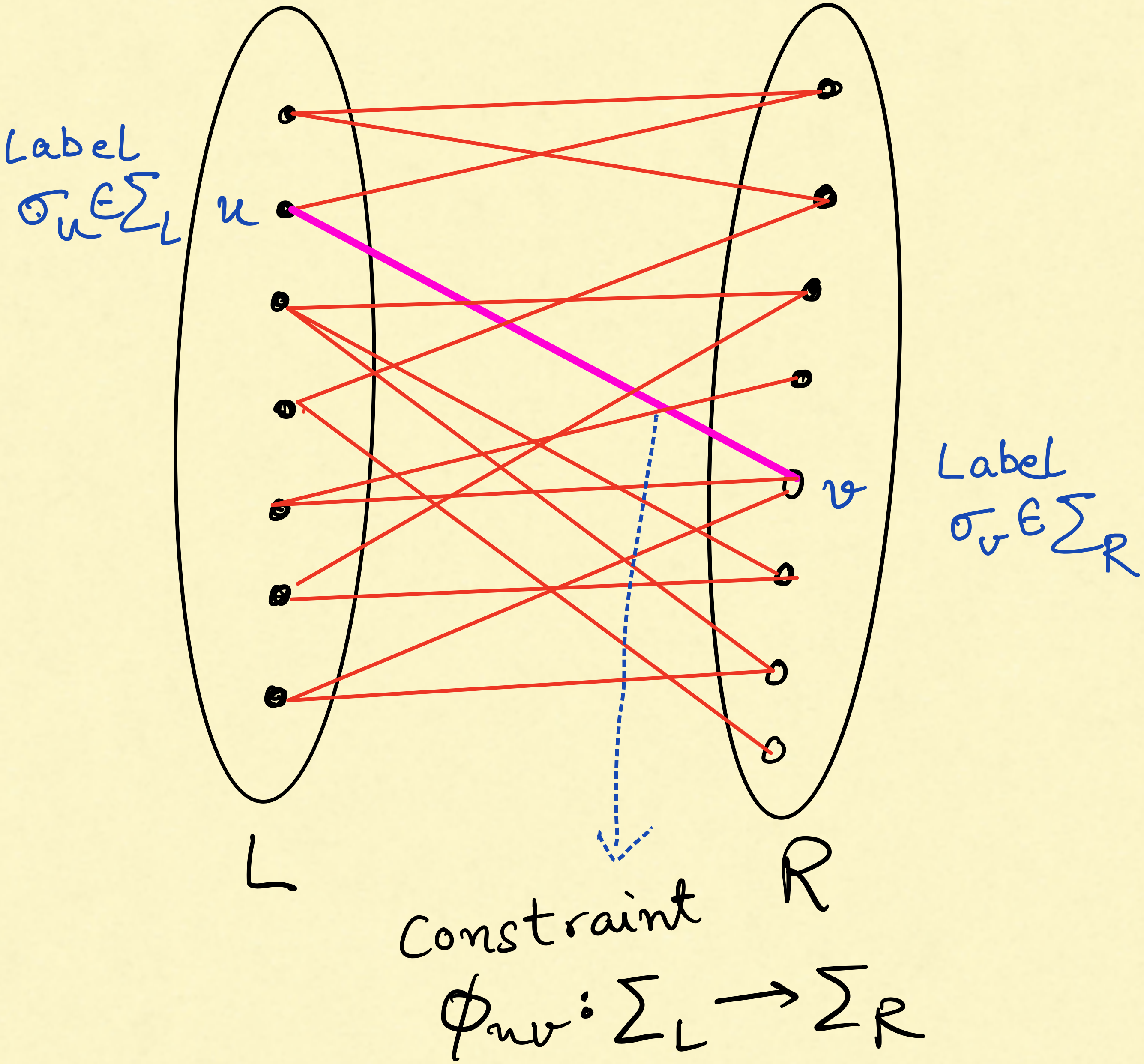

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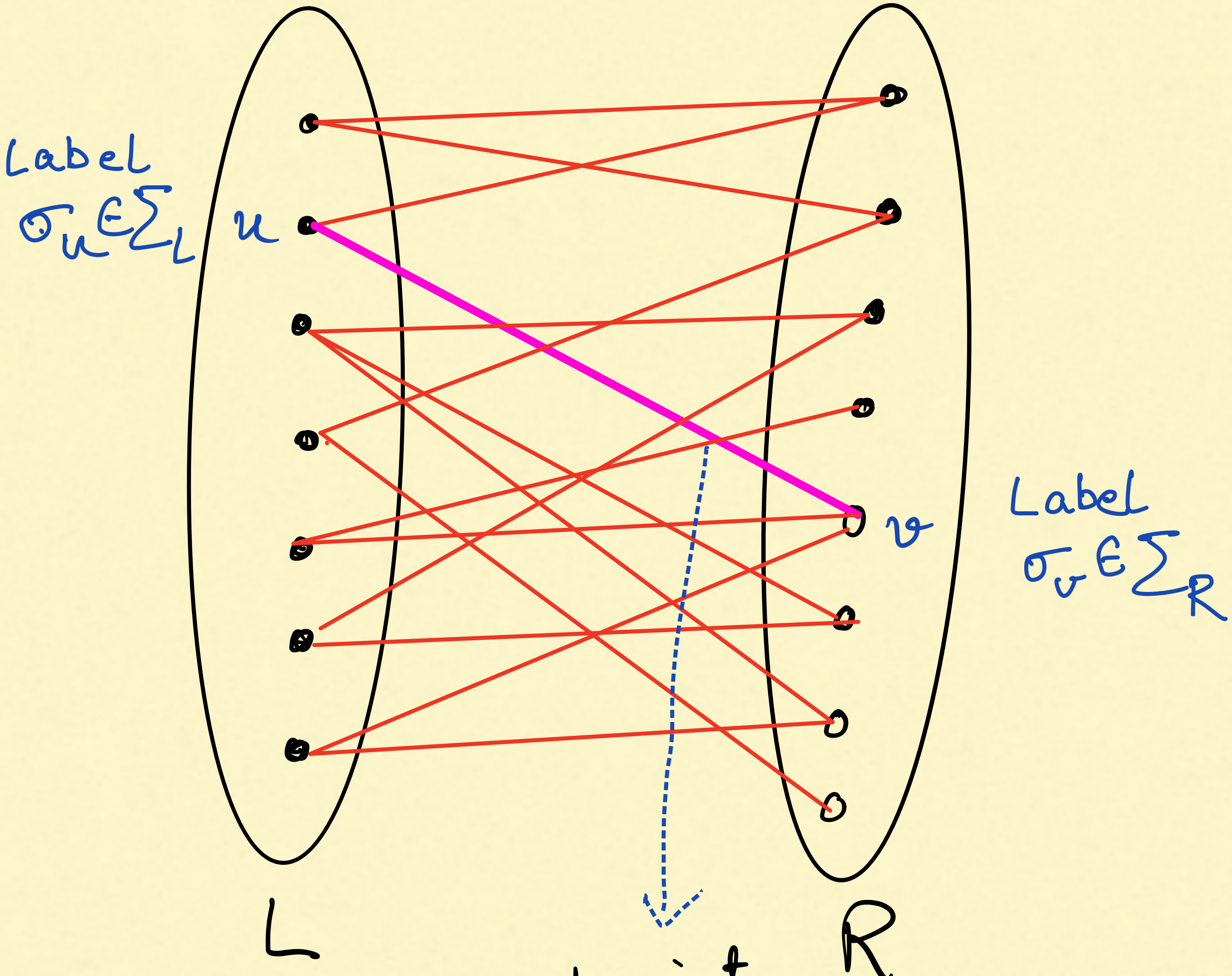

Constraint

$$\phi_{uv} : \Sigma_L \to \Sigma_R$$

Want $\phi_{uv}(\sigma_u) = \sigma_v$

$$\mathrm{val}_\psi(\sigma) = \underset{uv \sim E}{\mathbb{P}} \{ \phi_{uv}(\sigma_u) = \sigma_v \}$$

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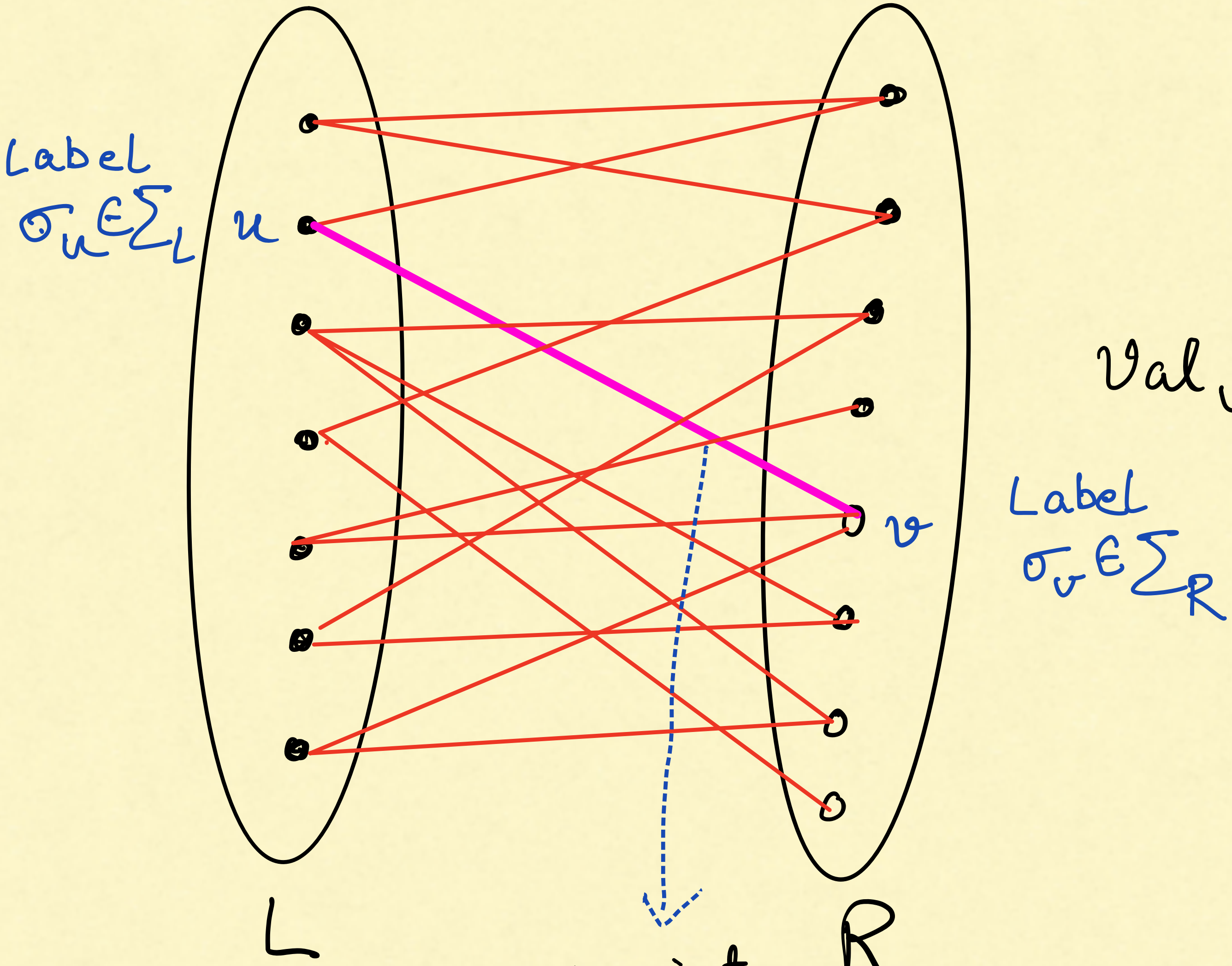

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$$val_{\psi}(A) = \underset{uv \sim E}{P}\{\phi_{uv}(A(u)) = A(v)\}$$

$$val(\psi) = \max_{\sigma} val_{\psi}(\sigma)$$

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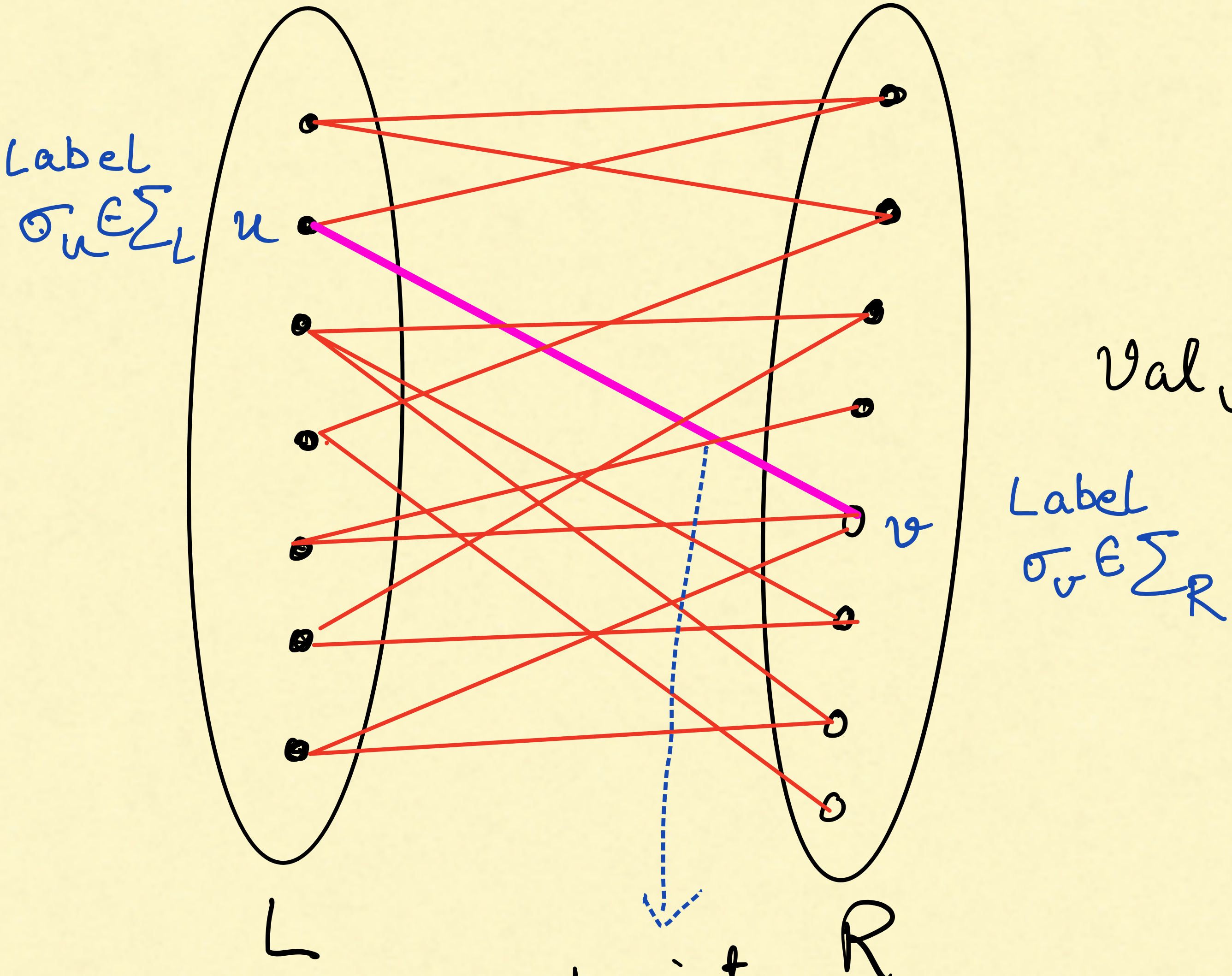

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Label Cover Problem

A bipartite graph: $G = (L \cup R, E)$

alphabets: Σ_L, Σ_R

For each $e \in E$,

constraint: $\phi_e : \Sigma_L \to \Sigma_R$

For assignments $A_L : L \to \Sigma_L$, $A_R : R \to \Sigma_R$

$$\mathrm{val}_\psi(A_L, A_R) = \underset{uv \sim E}{\mathbb{P}} \left\{ A_R(v) = A_L(\phi_{uv}(u)) \right\}$$

$$\mathrm{val}(\psi) = \max_{A_L A_R} \mathrm{val}_\psi(A_L, A_R).$$

Hardness of Label Cover

3SAT $\phi \implies$ Label Cover Ψ

with $\text{val}(\Psi) = \begin{cases} 1 & \text{if } \phi \text{ is sat} \\ \leq 1-\delta & \text{o.w.} \\ (\text{for some } \delta) & \end{cases}$

Hardness of Label Cover

$$3SAT\ \phi \xRightarrow{(i)} \text{Label Cover } \Psi_I$$

(or 2-CSP)

with $val(\Psi_I) = \begin{cases} 1 & \text{if } \phi \text{ is sat} \\ \leq 1-\delta & \text{o.w.} \\ (\text{for some } \delta) & \end{cases}$

(i) PCP Theorem

(ii) Gap Amplification

$$\Big\Downarrow (ii)$$

Label Cover Ψ

with $val(\Psi) = \begin{cases} 1 & \text{if } \phi \text{ is sat} \\ \leq \delta & \text{o.w.} \\ (\text{for any } \delta) & \end{cases}$

Hardness of 2-CSP

[Mei16] There is a $\delta \in (0,1)$ such that

there is a reduction from size n 3SAT ϕ to a 2-CSP Ψ over a regular graph with alphabet Σ s.t.

(i) $\text{size}(G) \leq n(\log n)^{O(\log\log n)}$ and $|\Sigma| = O(1)$

(ii) ϕ is satisfiable $\Rightarrow$ Ψ is satisfiable.

(iii) ϕ isn't satisfiable $\Rightarrow$ $\text{val}_{\Psi}(\Psi) \leq 1-\delta$.

Parallel Repitition

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$$G_t = (L^t \cup R^t, E_t) \text{ where}$$

$$(u_1 \ldots u_t) \sim (v_1 \ldots v_t) \text{ iff } u_i \sim v_i \ \forall i \in [t]$$

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$$\mathrm{val}(\psi) \leq 1-\varepsilon \Rightarrow \mathrm{val}(\psi^{\otimes t}) \leq (1-\varepsilon')^t$$

$$\varepsilon' = \mathrm{poly}(\varepsilon).$$

Agreement Tests on Sparse Families

Let $\{A_S : s \to \Sigma\}_{s \in \mathcal{F}}$ encode the assignment on each subset $s \in \mathcal{F} \subseteq \binom{V}{t}$.

$\mathcal{D}$ = distribution on $\mathcal{F} \times \mathcal{F}$. s.t.

$$\mathop{\mathbb{P}}_{s,s'}\left\{ A_S|_{s \cap s'} = A_{s'}|_{s' \cap s} \right\} \geq \varepsilon$$

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$\Rightarrow \exists\, G : V \to \Sigma$ s.t. $\Pr_{s}\left\{ \mathrm{dist}(G|_s, A_s) \leq \eta \right\} \geq \mathrm{poly}(\varepsilon)$.

Such a test is called ε-sound.

Agreement Test on our Complex

Fix N, ε. Then, there is a complex X with

(i) $|X(0)| \approx N$, degree $\leq 2^{\exp(\mathrm{poly}(1/\varepsilon))\sqrt{\log N}}$

(ii) for some $\ell = \exp(\mathrm{poly}(1/\varepsilon))$,

the distribution $\mathcal{D}$: $s, s' \sim \Pi_\ell$ conditioned on $|s \cap s'| = \sqrt{\ell}$

gives a ε-sound test.

Hardness of Label Cover

For any $\varepsilon \in (0,1)$, there exists a reduction from 3SAT instance ϕ of size n to labelcover Ψ of size $n' \leq n 2^{O(\sqrt{\log n \log\log n})}$ such that

1. ϕ is satisfiable $\Rightarrow$ Ψ is satisfiable.
2. ϕ is unsatisfiable $\Rightarrow$ $\text{val}(\Psi) \leq \varepsilon$.
3. $\Sigma_L = \Sigma^{\exp(O(1/\delta))}$, $\Sigma_R = \Sigma^{\exp(O(\sqrt{1/\delta}))}$
$$|\Sigma| \leq 2^{O(\sqrt{\log n \log\log n})}$$

Thanks!