Linearly Ordered (LO) Coloring of 2-LO Colorable 3-Uniform Hypergraphs

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Coloring Problems

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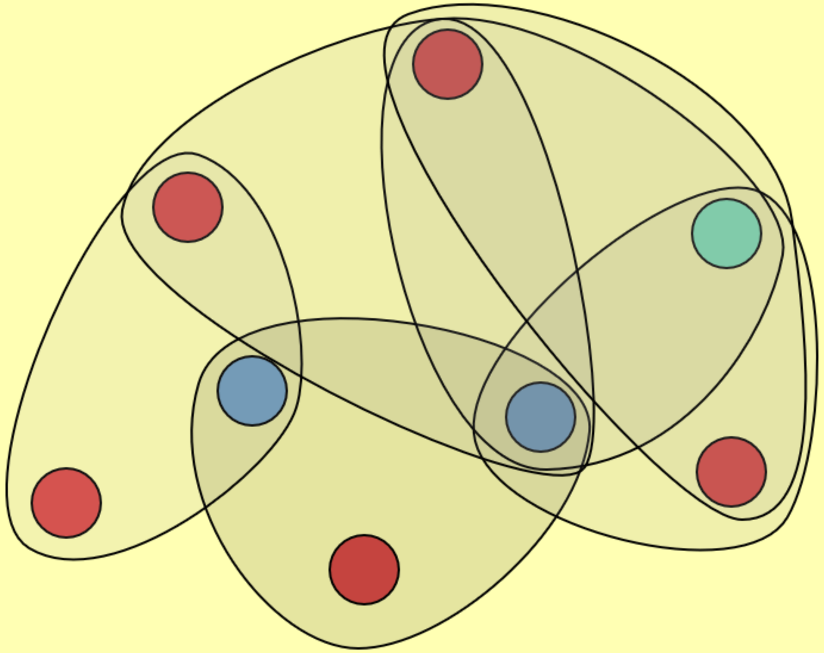

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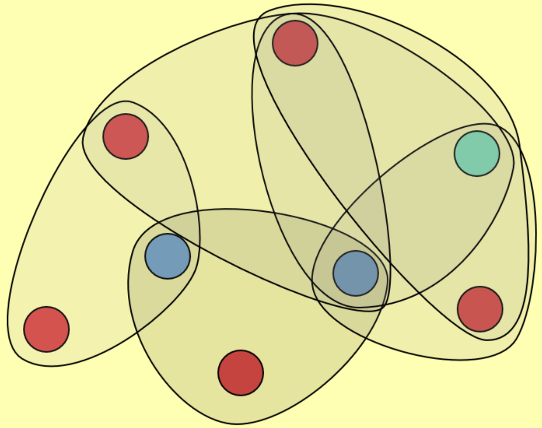
- General graphs ⟶ hardness of approximation



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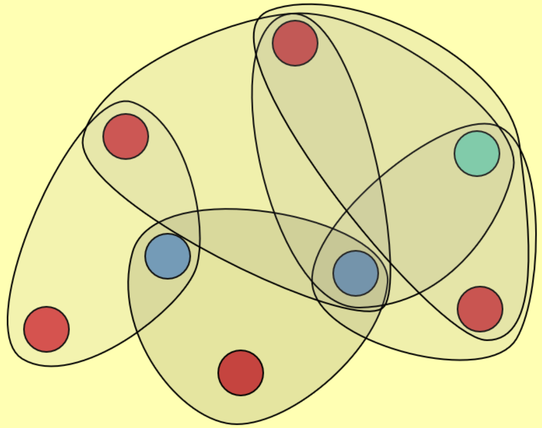

- General graphs → hardness of approximation

- No polynomial-time algorithm for coloring a graph using $n^{1-\varepsilon}$ colors unless NP=P (Fiege, Killian, J.Comp'98; Zuckerman, ToC'07).

Approximate Coloring Problems

- Given a 3-colorable graph find a coloring with minimum number of colors.

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- Given a 3-colorable graph find a coloring with minimum number of colors.

- Given a 2-colorable 3-uniform hypergraph find a coloring with minimum number of colors.

Approximate LO Coloring of a Hypergraph

- Given a hypergraph assign each vertex a color from a linearly ordered set of colors so that each edge has a unique maximum.

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- Given a hypergraph assign each vertex a color from $\{1,2,3,\dots\}$ so that each edge has a unique maximum.

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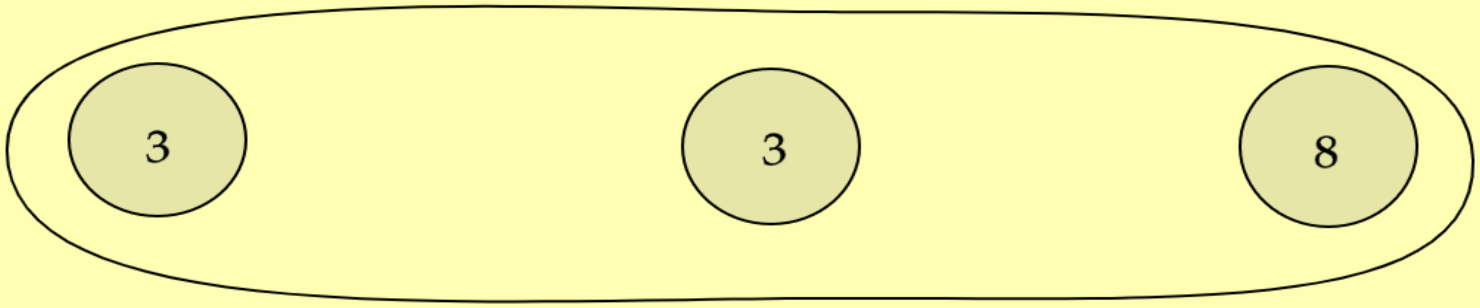

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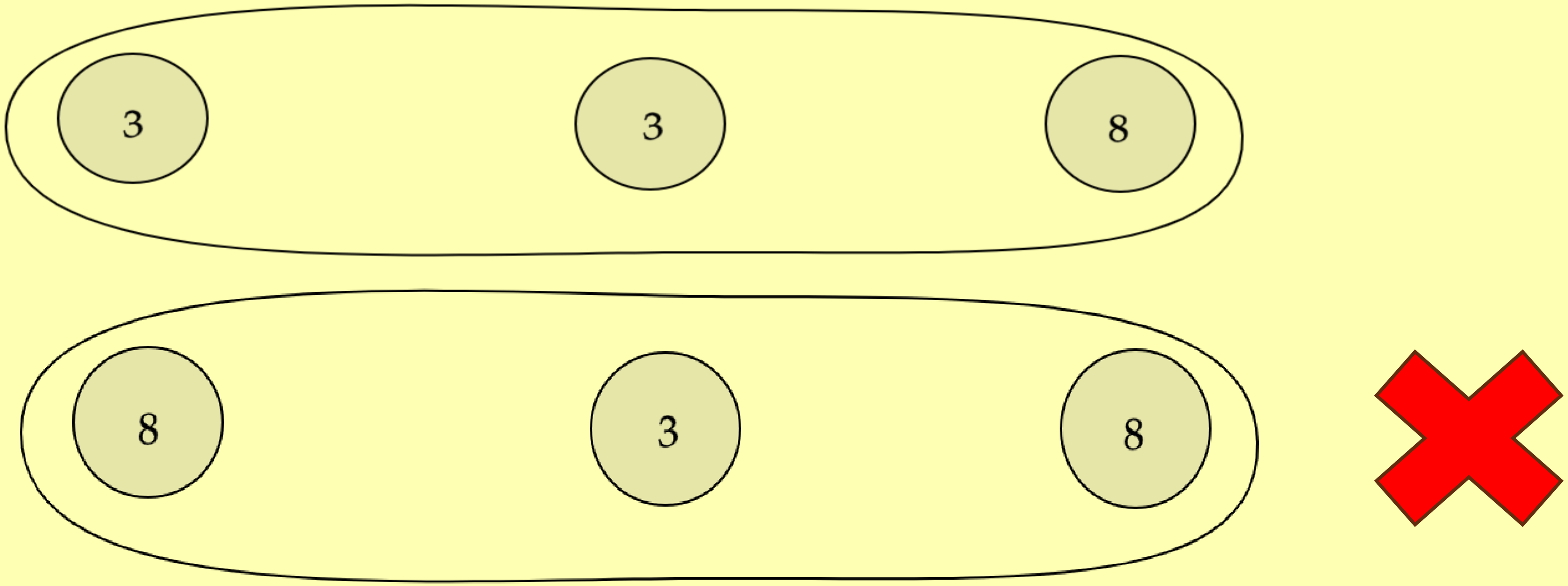
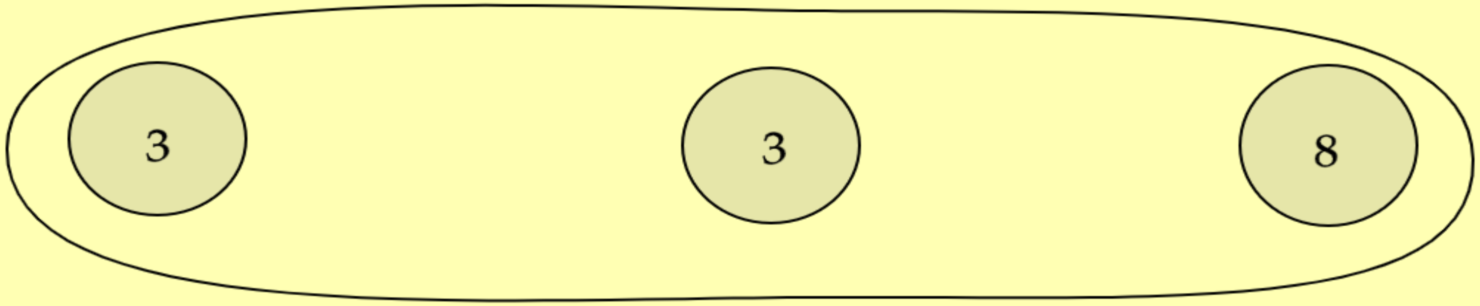


Approximate LO Coloring of a Hypergraph

- Given a hypergraph assign each vertex a color from {1,2,3, … } so that each edge has a unique maximum.

- Given a 2-LO colorable 3-uniform hypergraph find a coloring using fewest possible colors.



Why look at LO Coloring of a Hypergraph?

- Approximate coloring problems → Promise Constraint Satisfaction Problems (PCSPs).

- Barto, Battistelli, and Berg [STACS'21]:

an almost complete characterization of the tractability for PCSPs 3-uniform hypergraph with 2 colors under various notions of coloring.

- LO coloring was the only gap in their characterization.

Results for 2-LO colorable 3-uniform hypergraphs

- Previous Result (Nakajima, Živný-TCT'22): LO coloring using at most $\tilde{O}(n^{1/3})$ colors.

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- Previous Result (Nakajima, Živný-TCT'22): LO coloring using at most $\tilde{O}(n^{1/3})$ colors.
- Our result: LO coloring using at most $\tilde{O}(n^{1/5})$.
- Concurrent Work (Håstad, Martinsson, Nakajima, Živný-APPROX'24): LO coloring using at most $2\log_2 n$ colors.

Background

Background: SDPs

- Semidefinite Programing (SDP) problems are optimization problems:
  - Objective: linear
  - Constraints: (a) linear constraints (b)psd-ness constraint.

Example:

$$\max \sum_{i,j} c_{ij} x_{ij}$$

$$\sum_{i,j} a_{ijk} x_{ij} = b_k \qquad \forall k$$

$$X = (x_{ij}) \succcurlyeq 0$$

Background: VPs

- Vector Programing (VP) problems are optimization problems involving n-dimensional vectors.
  - Objective: linear in inner-products
  - Constraints: linear in the inner-product.

Example:

$$\max \sum_{i,j} c_{ij} \langle v_i, v_j \rangle$$

$$\sum_{i,j} a_{ijk} \langle v_i, v_j \rangle = b_k \qquad \forall k$$

$$v \in \mathbb{R}^n$$

Background: SDP and VP

- Fact: VPs and SDPs are equivalent.

- It is easier to deal with VPs.

- VPs are referred to as SDPs as well.

"Proof" of the result

Integer Program for 2-LO colorable hypergraphs

Mapping the colors $1 \mapsto -1$ and $2 \mapsto 1$ we get:

$$x_i + x_j + x_k = -1 \qquad \forall \{i, j, k\} \in E$$

$$x_i \in \{-1, +1\} \qquad \forall i \in V$$

SDP relaxation for 2-LO colorable hypergraphs

$$v_i + v_j + v_k = -v_0 \qquad \forall \{i, j, k\} \in E$$

$$\|v_i\|^2 = 1 \qquad \forall i \in V \cup \{0\}$$

$$v_i \in \mathbb{R}^{n+1} \qquad \forall i \in V \cup \{0\}$$

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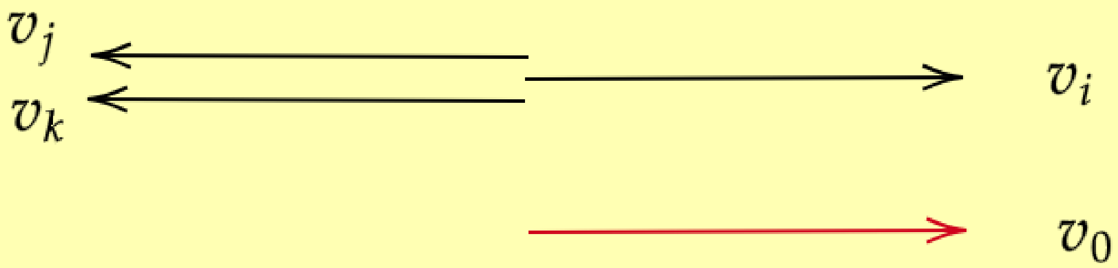

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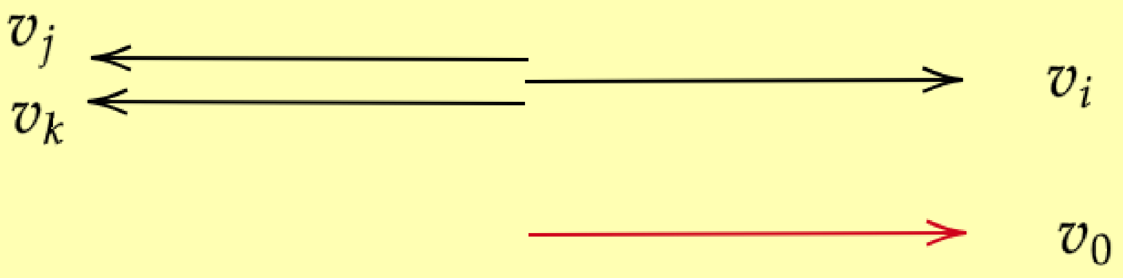

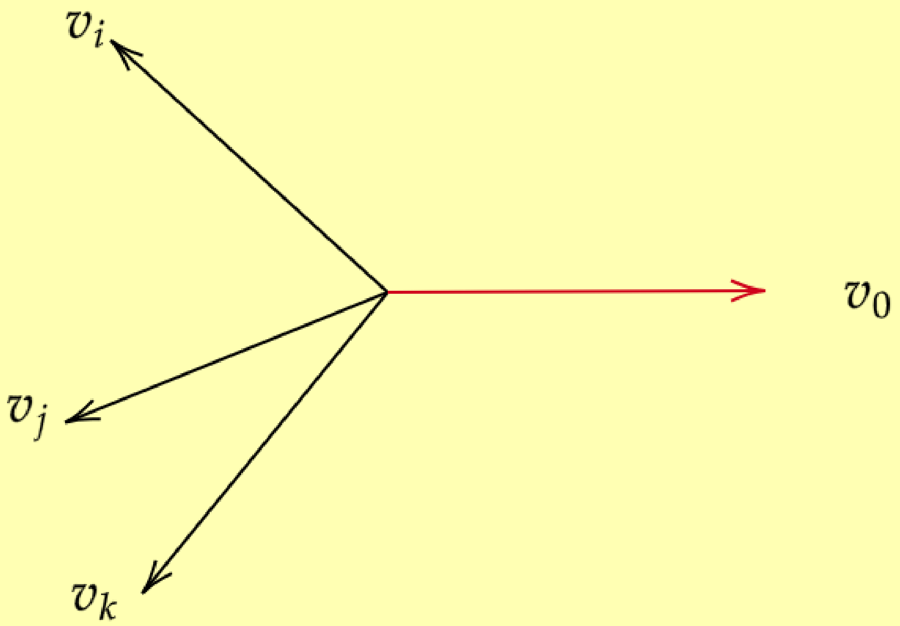

Coloring by Finding Independent Sets

- Most coloring algorithms proceed by iteratively coloring a 'large' independent set.
- For hypergraphs, there are possibly many ways to define independent set.
- The standard notion of independent set for hypergraphs is not useful here.

Odd and Even Independent Sets

- For LO coloring the following notion of independent sets is useful.
- Odd independent set: $S \subseteq V$ is an odd independent set if $|S \cap e| \leq 1$ for each edge e.
- Even independent set: $S \subseteq V$ is an even independent set if $|S \cap e| \in \{0, 2\}$ for each edge e.

Combinatorial Rounding

- The ideal solution of the SDP would be 1-dimensional.

- The values $\gamma_a = \langle v_a, v_0 \rangle$ contain a lot of information about the color that can be assigned to a if $|\gamma_a| \approx 1$.

Key Observations

- Observation 1. If $\{a, b, c\}$ is an edge, then $\gamma_a + \gamma_b + \gamma_c = -1$.
- Observation 2. For each vertex a, we have $|\gamma_a| \leq 1$.

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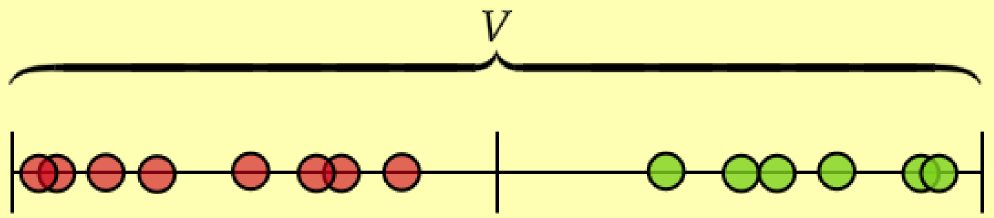
- Observation 2. For each vertex a, we have $|\gamma_a| \leq 1$.
- Proof. Cauchy-Schwarz!!

First Iteration



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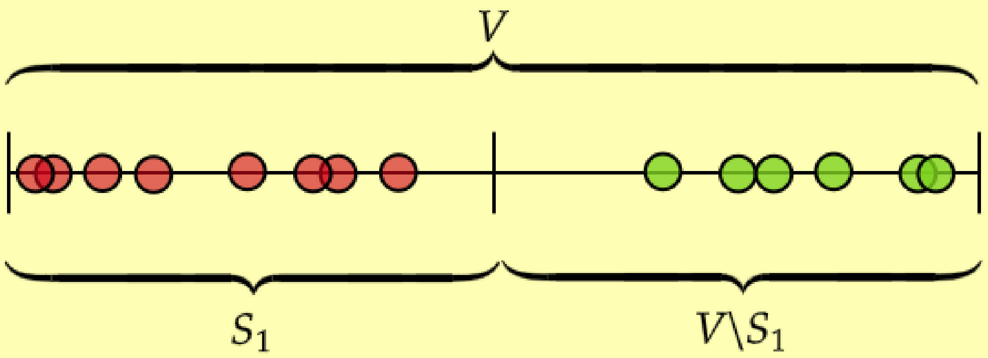

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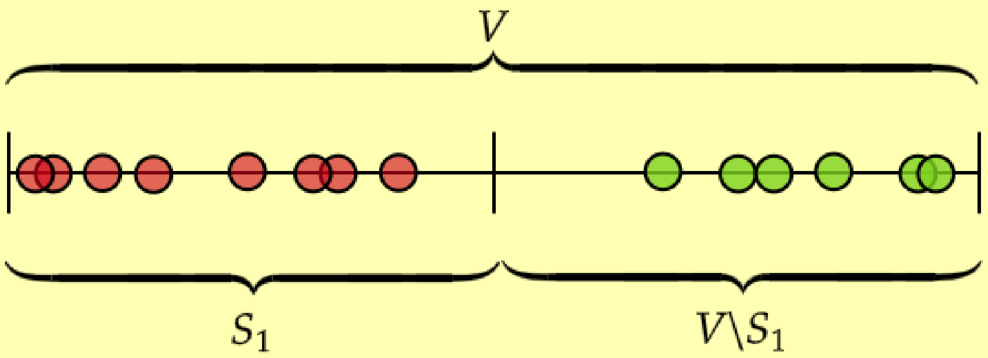

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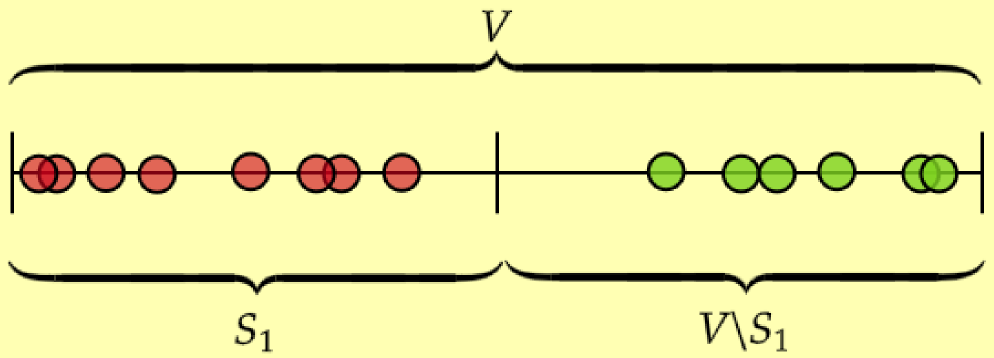
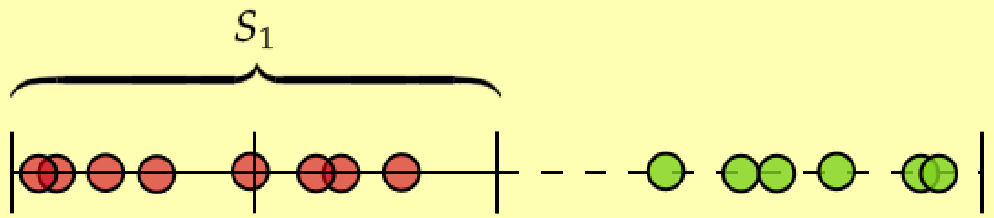


- The set $V \setminus S_1$ is an odd independent set.

- Use the largest color to color it.

Second Iteration



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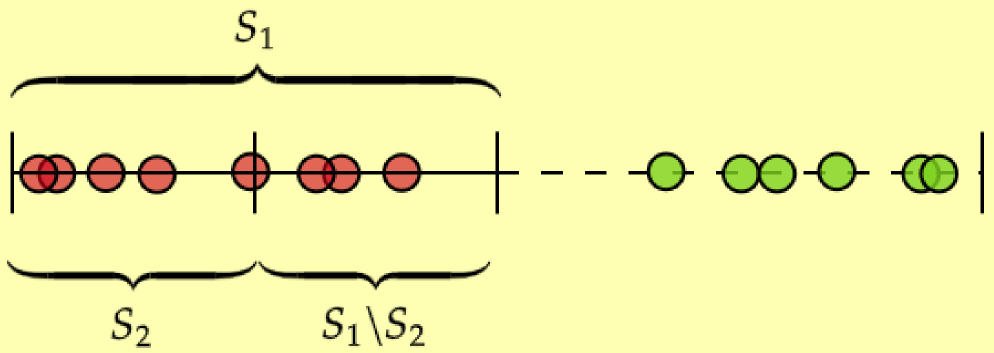

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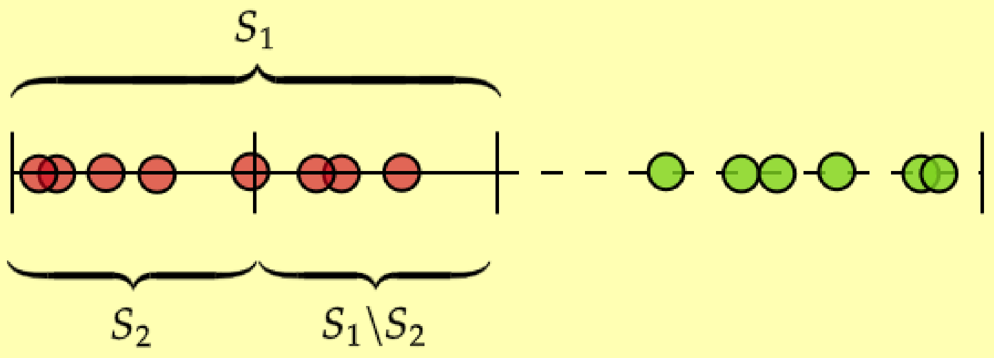

- The set $S_1 \setminus S_2$ is an odd independent set in the hypergraph induced by S_1.

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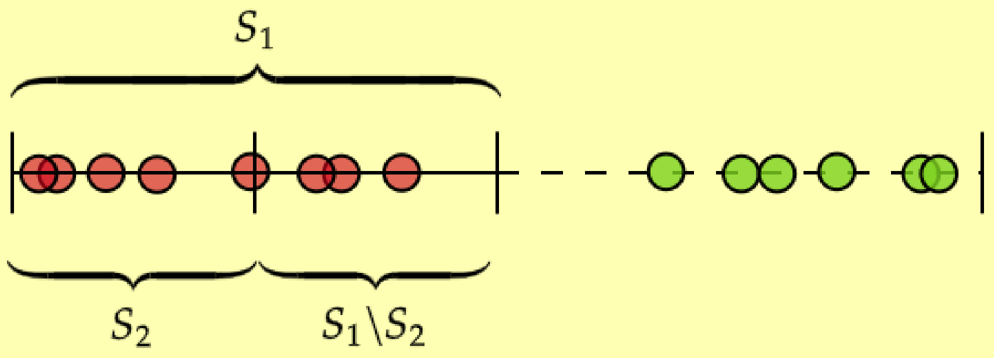

- The set $S_1 \setminus S_2$ is an odd independent set in the hypergraph induced by S_1.
- Use the second largest color to color it.

And so on...

The performance guarantee

- We have $I_j \approx [-\frac{1}{3} - \varepsilon_j, -\frac{1}{3} + \varepsilon_j]$ and corresponding set $S_j = \{a \in V \mid \gamma_a \in I_j\}$ with $\varepsilon_j = \frac{1}{2^j}$.

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- After $O(\log 1/\varepsilon)$ all the vertices remaining have $\gamma \approx -\frac{1}{3}$.

Handling the balanced case

- We used standard Hyperplane rounding for the balanced case to obtain our result.

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- A (slight) random perturbation to the remaining vectors + combinatorial rounding = $O(\log n)$ coloring.

Some interesting directions

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- Can the known methods be generalized to 2-LO colorable r-uniform hypergraphs?

Questions?

Thanks