Sparse Cuts in Hypergraphs from Random Walks on Simplicial Complexes

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- Relate hypergraph expansion and link expansion of a simplicial complex.

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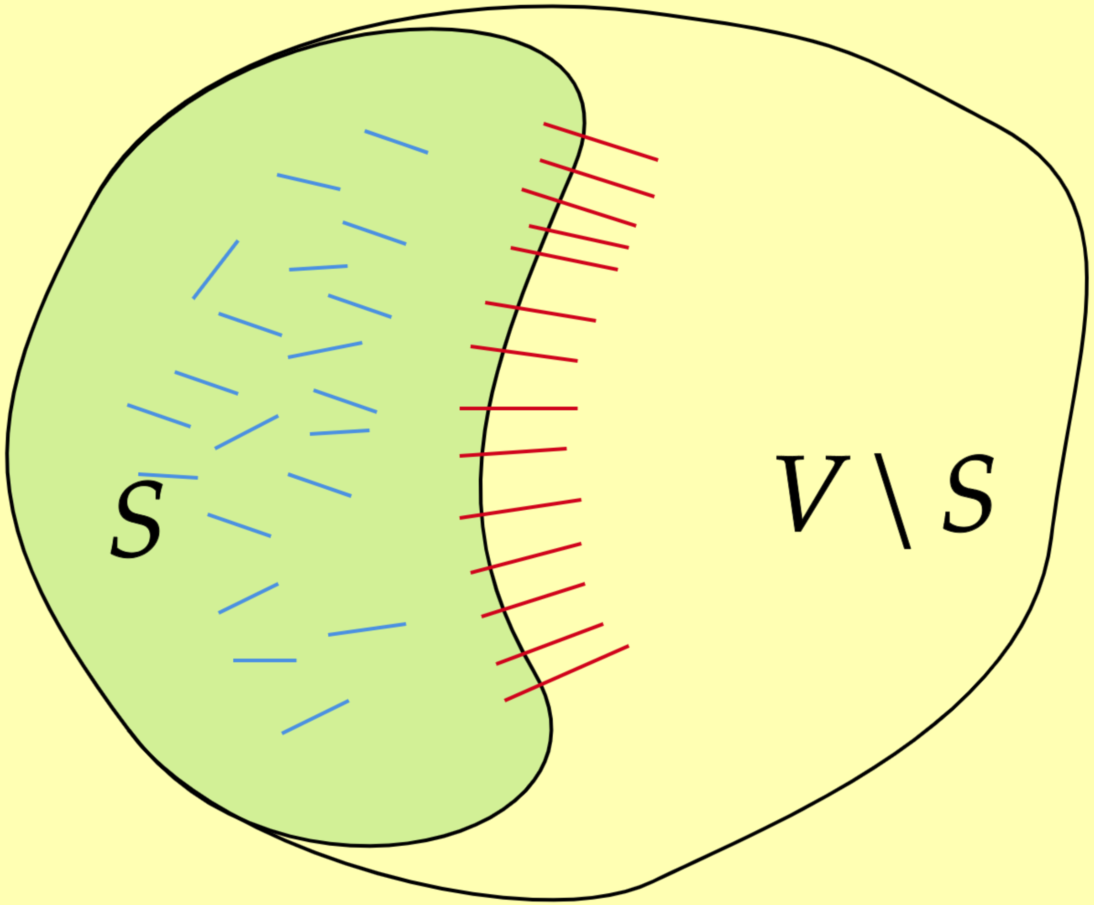

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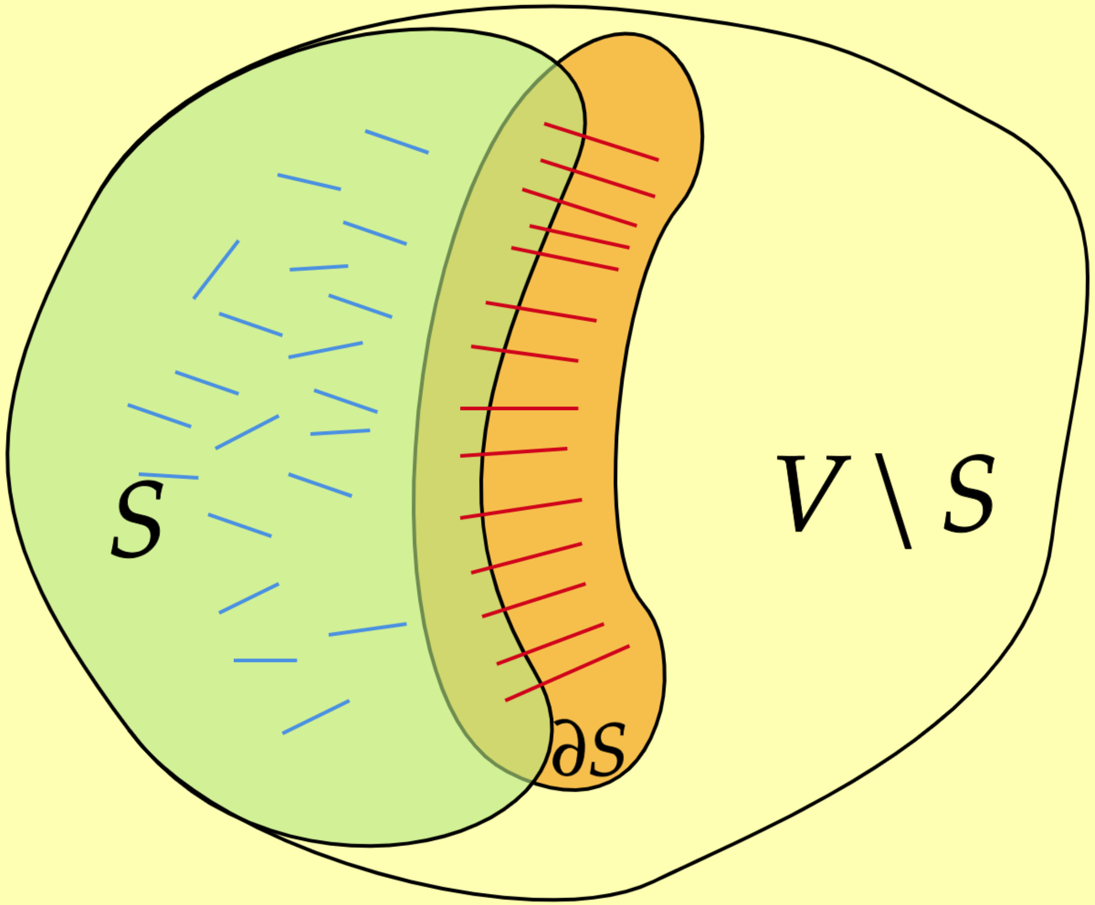

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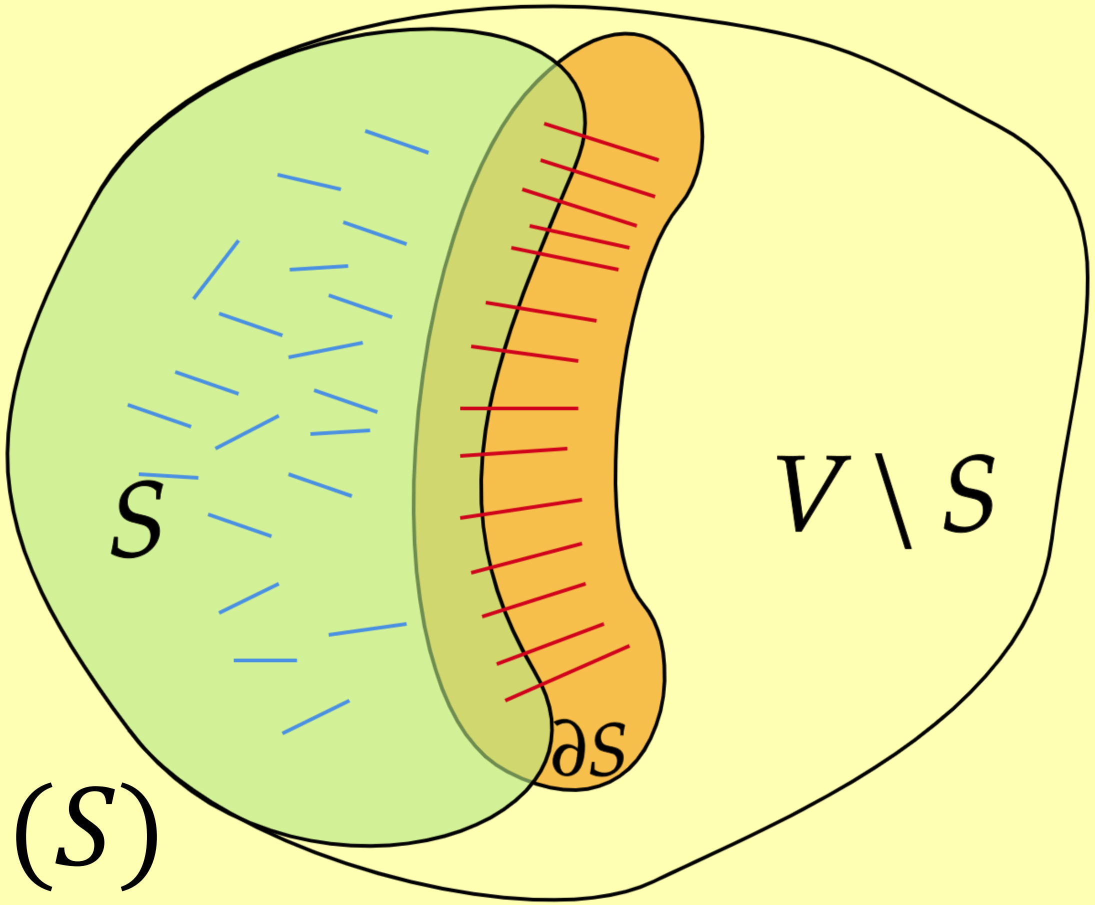

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Hypergraph Expansion

$$\phi_H(S) = \frac{w(\partial(S))}{vol(S)}$$

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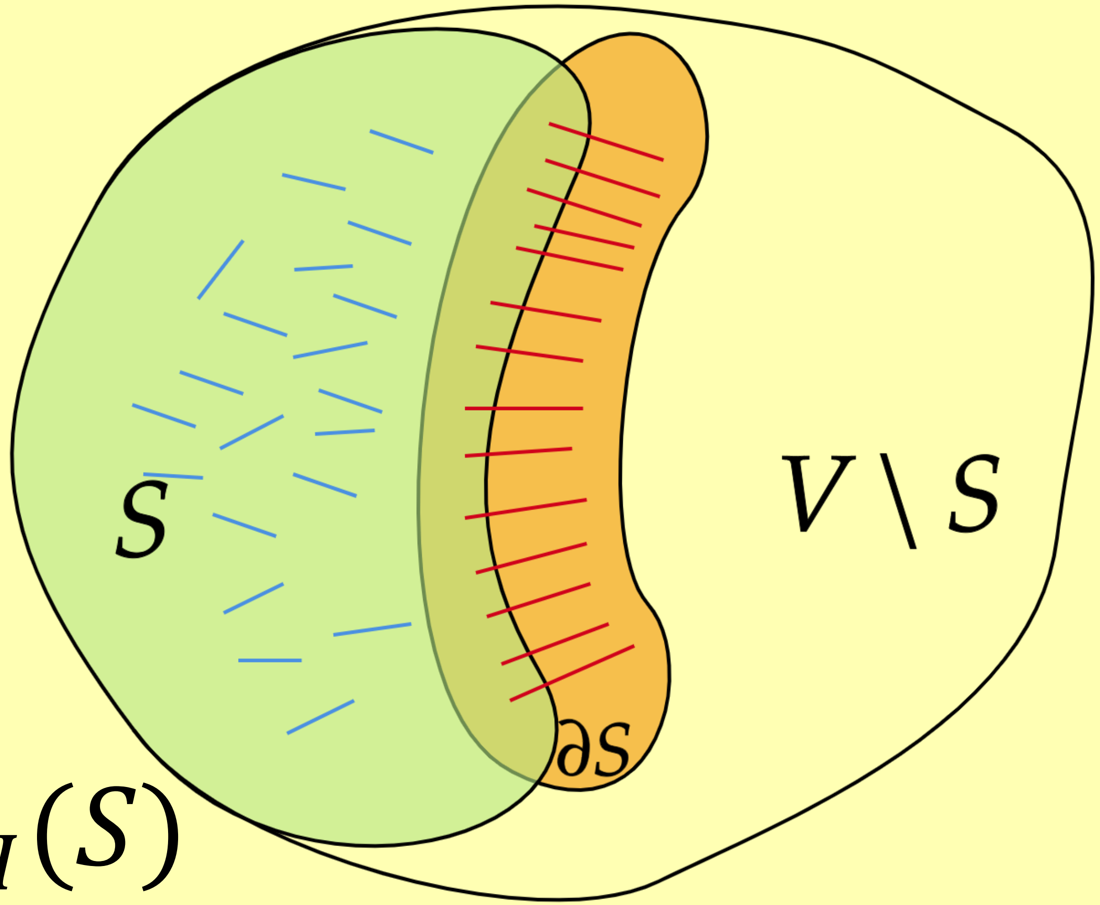

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What about hypergraphs?

The Spectra of a Hypergraph

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- It is however not known how to compute such eigenvalue above operator as it is non-linear (in general).
- Instead, here we try to use eigenvalues of random walks on simplicial complexes to do so.

Simplicial Complexes

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  The dimension of the largest faces is the dimension of the complex.
- We use $X(l)$ for the collection of the l-dimensional faces.

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- Use Π_l to denote (normalized) weights on the faces of dimension l.
- The weight of the top faces can be arbitrary, i.e., Π_k is arbitrary.
- The weight of smaller faces is essentially the weighted "degree", i.e.,

$$\Pi_l(s) = \frac{\sum_{e \supseteq s} \Pi_k(e)}{\binom{k}{l}}.$$

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Note that this determines the other weights as well.

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- Solving CSPs:
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Random Walks on Simplicial Comlexes

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- In a Down Walk one goes from a face $\sigma \in X(m)$ to $\tau \in X(l)$ with probability $\frac{\Pi_l(\tau)}{\Pi_m(\sigma)}$.
- An Up-Down Walk on l passing through m is constructed by combining the up and down walks above.

Swap Walks

- In a swap walk from l to m we move from a face $\sigma \in X(l)$ to $\tau \in X(m)$ with probability $\frac{\Pi_{l+m}(\sigma \sqcup \tau)}{\Pi_l(\sigma)}$.

Motivating Problem: Constraint Satisfaction Problems

Examples

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- Given linear equations modulo p with 3 variables each maximize the number of equations satisfied.

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- If all constraints are k-ary, we call the corresponding instances k-CSP instances.

Known Results on solving CSPs

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- Alev, Jeronimo, Tulsiani (FOCS'19) gave a similar algorithm to solve k-CSPs on instances splittable hypergraphs.
- The notion of splittability is a natural analogue of threshold rank for hypergraph based on threshold ranks of swap walk on the corresponding simplicial complex.

Can we use random walks on simplicial complexes to compute sparse cuts in hypergraph?

We show there are expanding hypergraphs with large threshold rank swap walks.

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- Also, swap walks between dimensions 1 and $k-1$ also have threshold rank $\Omega_k(n)$.

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This means we cannot compute a decomposition of non-splittable H with each component being splittable.

We show there are expanding hypergraphs with large threshold rank up-down walks.

Interestingly, the up-down walk between dimensions larger than 2 on the same hypergraph also have large threshold rank.

High Dimensional Expanders

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- In fact, all the paper mentioned before used it in some capacity.
- So, it is again natural to ask whether the notions of hypergraph expansion and link expansion (of the corresponding simplicial complex) related.

There is an expanding hypergraph with small link expansion

There are k-uniform hypergraph H with at least n vertices such that

- $\phi_H \geq \frac{1}{(3k)^k}$, in other words, the expansion is large

Again, we show there are expanding hypergraph with small link expansion

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- $\phi_H \geq \frac{1}{(3k)^k}$, in other words, the expansion is large, and
- link expansion is $\Theta\left(\frac{1}{n^2}\right)$.

Questions?

Thanks